

$$g'(\theta) = \frac{d}{d\theta} \frac{2 \log(1+\theta)}{1+\theta} = \frac{-2}{(1+\theta)^2} \log(1+\theta) + \frac{2}{(1+\theta)^2}$$

↑  
product rule

Math 582

Quiz 2 2022

Name \_\_\_\_\_

1) Suppose that  $X_1, \dots, X_n$  are iid  $N(\mu, \sigma^2)$ .

a) Find the limiting distribution of  $\sqrt{n} (\bar{X}_n - \mu)$ .  $\rightarrow N(0, \sigma^2)$

by CLT

b) Let  $g(\theta) = [\log(1+\theta)]^2$ . Find the limiting distribution of  $\sqrt{n} (g(\bar{X}_n) - g(\mu))$  for  $\mu > 0$ .

$$g'(\theta) = \frac{2 \log(1+\theta)}{1+\theta}$$

$$\text{so } \sqrt{n} (g(\bar{X}_n) - g(\mu)) \xrightarrow{D}$$

$$N\left[0, \left[\frac{2 \log(1+\mu)}{1+\mu}\right]^2 \sigma^2\right] \sim N\left(0, \left[\frac{2 \log(1+\mu)}{1+\mu}\right]^2 \sigma^2\right)$$

$$\sim N\left[0, \frac{4 [\log(1+\mu)]^2 \sigma^2}{(1+\mu)^2}\right]$$

c) Let  $g(\theta) = [\log(1+\theta)]^2$ . Find the limiting distribution of  $n (g(\bar{X}_n) - g(\mu))$  for  $\mu = 0$ . Hint: use the Second Order Delta Method.

$$g(0) = 0 \quad g'(0) = 0 \quad \text{since } \log(1) = 0$$

$$g''(\theta) = \frac{d}{d\theta} \frac{2 \log(1+\theta)}{1+\theta} = \frac{(1+\theta)^{-2} \cdot 2 - 2 \log(1+\theta)}{(1+\theta)^2}$$

$$\left(\frac{f}{g}\right)' = \frac{df - f dg}{g^2} \quad \leftarrow \text{quotient rule} \quad \frac{2}{(1+\theta)^2}$$

$$= \frac{2 [1 - \log(1+\theta)]}{(1+\theta)^2}, \quad g''(0) = \frac{2}{1} = 2$$

$$\text{so } n [g(\bar{X}_n) - g(\mu)] \xrightarrow{D} \frac{1}{2} \sigma^2 g''(0) \chi^2_1$$

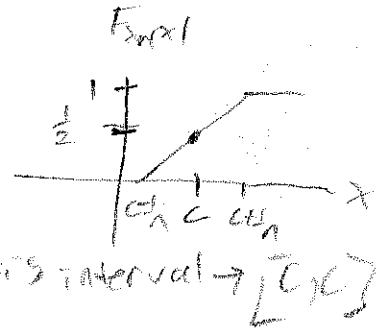
$$\sim \frac{1}{2} \sigma^2 2 \chi^2_1 \quad \sim \sigma^2 \chi^2_1$$

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$$X_n \sim U(c - \frac{1}{n}, c + \frac{1}{n})$$

2) Suppose

$$F_{X_n}(x) = \begin{cases} 0, & x \leq c - \frac{1}{n} \\ \frac{n}{2}(x - c + \frac{1}{n}), & c - \frac{1}{n} < x < c + \frac{1}{n} \\ 1, & x \geq c + \frac{1}{n} \end{cases}$$



Does  $X_n \xrightarrow{D} X$  for some random variable  $X$ ? Prove or disprove. If  $X_n \xrightarrow{D} X$ , find  $X$ .

$$F_{X_n}(x) \rightarrow G(x) = \begin{cases} 0 & x < c \\ \frac{1}{2} & x = c \\ 1 & x > c \end{cases}$$

$\therefore X_n \xrightarrow{D} X$  where  $P(X=c) = 1$

$$\text{and } F_X(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

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PM

3) Suppose  $Y_n \sim \text{EXP}(n)$  with cdf  $F_{Y_n}(y) = 1 - \exp(-y/n)$  for  $y \geq 0$  and  $F_{Y_n}(y) = 0$  for  $y < 0$ . Does  $Y_n \xrightarrow{D} Y$  for some random variable  $Y$ ? Prove or disprove. If  $Y_n \xrightarrow{D} Y$ , find  $Y$ .

$$e^{-\frac{y}{n}} \rightarrow e^{-0} = 1 \quad \forall y \geq 0$$

$$\text{So } F_{Y_n}(y) \rightarrow 0 = G(y) \quad \forall y$$

$\therefore Y_n$  does not converge in distribution to any RV  $Y$  since  $G(y)$  is continuous but not a cdf.

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