Math 582 Quiz 3 2022 Name____________

pm

1) Suppose $Y_1, ..., Y_n$ are iid $POIS(\theta$. Then the MLE of θ is $\hat{\theta}_n = \overline{Y}_n$, and $I_1(\theta) = 1/\theta$.
a) Find the limiting distribution of $\sqrt{n}(\overline{Y}_n - c)$ for appropriate constant c.

$\sqrt{n}(\overline{Y} - \theta) \xrightarrow{D} N(0, \theta)$ by CLT

b) Let $\tau(\theta) = \theta^2$. Find the limiting distribution of $\sqrt{n}[\tau(\hat{\theta}_n) - \tau(\theta)]$ using the Standard Limit Theorem or the Delta Method.

$\tau'(\theta) = 2\theta$ and $[\tau'(\theta)]^2 = 4\theta^2$

$\therefore \sqrt{n}((\overline{Y}_n)^2 - \theta^2) = \sqrt{n}[\tau(\hat{\theta}_n) - \tau(\theta)] \xrightarrow{D} \underline{N(0, 4\theta^3)}$

$\dfrac{[\tau'(\theta)]^2}{I_1(\theta)} = \dfrac{4\theta^2}{(1/\theta)} = 4\theta^3$

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2) Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y) = \frac{\theta}{y^2} \exp\left(\frac{-\theta}{y}\right)$$

where $y > 0$ and $\theta > 0$. Then $MED(Y) = \theta/\log(2)$. Find the limiting distribution of $\sqrt{n}(MED(n) - MED(Y))$.

$\xrightarrow{D} N\left(0, \dfrac{1}{4[f(MED(Y))]^2}\right)$

$f(MED(Y)) = \dfrac{\theta [\log(2)]^2}{\theta^2} \exp\left(\dfrac{-\theta \log(2)}{\theta}\right)$

$= \dfrac{[\log(2)]^2}{2\theta}$ $\therefore 4[f(MED(Y))]^2 = \dfrac{4[\log(2)]^4}{4\theta^2} = \dfrac{[\log(2)]^4}{\theta^2}$

so $\sqrt{n}\Big(MED(n) - \underbrace{\dfrac{\theta}{\log(2)}}_{MED(Y)}\Big) \xrightarrow{D} N\left(0, \dfrac{\theta^2}{[\log(2)]^4}\right)$

$\sim N(0, 4.33210\theta^2)$

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$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \mu)^2$ $\quad W_i = (Y_i - \mu)^2 \quad \frac{W_i}{\sigma^2} \sim \chi_1^2$

$E(W_i) = \sigma^2 (1) \quad V(W_i) = \sigma^4 2$

3) Let $Y_1, ..., Y_n$ be iid $N(\mu, \sigma^2)$ with μ known. Let $\hat{\sigma}_n^2$ be the MLE of σ^2 with $I_1(\sigma^2) = \dfrac{1}{2\sigma^4}$.

a) Find the limiting distribution of $\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2)$. $\xrightarrow{D} N\left(0, \frac{1}{I_1(\sigma^2)}\right)$

$\sim N(0, 2\sigma^4)$

b) Find the limiting distribution of $\sqrt{n}[\sqrt{\hat{\sigma}_n^2} - \sigma]$. Note that $\tau(\sigma^2) = \sqrt{\sigma^2}$. Taking $\theta = \sigma^2$ could be useful.

$\tau(\theta) = \sqrt{\theta} = \theta^{\frac{1}{2}} \qquad \tau'(\theta) = \frac{1}{2}\theta^{-\frac{1}{2}} = \frac{1}{2\sqrt{\theta}}$

$\tau'(\sigma^2) = \frac{1}{2\sqrt{\sigma^2}} = \frac{1}{2\sigma} \qquad = \frac{d\tau(\theta)}{d(\sigma^2)}$

$\frac{[\tau'(\sigma^2)]^2}{I_1(\sigma^2)} = [\tau'(\sigma^2)]^2 \, 2\sigma^4 = \frac{2\sigma^4}{4\sigma^2} = \frac{\sigma^2}{2}$

$\therefore \sqrt{n}\left[\sqrt{\hat{\sigma}_n^2} - \sigma\right] \xrightarrow{D} N\left(0, \frac{\sigma^2}{2}\right)$

Standard Limit Th or Delta Method works

$\downarrow$

$[\tau'(\sigma^2)]^2 \, 2\sigma^4 = \frac{2\sigma^4}{4\sigma^2} = \frac{\sigma^2}{2}$

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