

take home

1) Suppose $\mathbf{X}_1, \dots, \mathbf{X}_n$ are iid $k \times 1$ random vectors where $E(\mathbf{X}_i) = (\mu_1, \dots, \mu_k)^T$ and $Cov(\mathbf{X}_i) = \text{diag}(\sigma_1^2, \dots, \sigma_k^2)$, a diagonal $k \times k$ matrix with j th diagonal entry σ_j^2 . The nondiagonal entries are 0. Find the limiting distribution of $\sqrt{n}(\bar{\mathbf{X}} - \mathbf{c})$ for appropriate vector \mathbf{c} .

$$\sqrt{n} \left[\bar{\mathbf{X}} - \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} \right] \xrightarrow{D} N_k \left[\mathbf{0}, \text{diag}(\sigma_1^2, \dots, \sigma_k^2) \right]$$

$$\sqrt{n} \left[\bar{\mathbf{X}} - E(\mathbf{X}_i) \right] \xrightarrow{D} N_k \left[\mathbf{0}; \text{Cov}(\mathbf{X}_i) \right]$$

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2) Suppose $Y_n \xrightarrow{P} Y$. Then $W_n = Y_n - Y \xrightarrow{P} 0$. Define $X_n = Y$ for all n . Then $X_n \xrightarrow{D} Y$. Then $Y_n = X_n + W_n \xrightarrow{D} Z$ by Slutsky's Theorem. What is Z ?

$$X_n \xrightarrow{D} X = Y \quad W_n \xrightarrow{P} 0 \quad \text{given}$$

$$\text{So } \underbrace{X_n}_{\xrightarrow{D} Y} + \underbrace{W_n}_{\xrightarrow{P} 0} \xrightarrow{D} X + 0 = Y + 0 = \boxed{Y = Z}$$

$$\underbrace{\hspace{10em}}_{X_n + Y_n - Y}$$

(Proves $Y_n \xrightarrow{D} Y$ if $Y_n \xrightarrow{P} Y$ using Slutsky's theorem Lehmann p 71.)

$$(Y_n \xrightarrow{P} Y \Rightarrow Y_n \xrightarrow{D} Y)$$

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pm

3) The method of moments estimator for $Cov(X, Y) = \sigma_{X,Y}$ is

$$\hat{\sigma}_{X,Y} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}). \text{ Another common estimator is}$$

$$S_{X,Y} = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) = \frac{n}{n-1} \hat{\sigma}_{X,Y}. \text{ Using the fact that } \hat{\sigma}_{X,Y} \xrightarrow{P} \sigma_{X,Y} \text{ when}$$

the covariance exists, prove that $S_{X,Y} \xrightarrow{P} \sigma_{X,Y}$ with Slutsky's Theorem. Hint: $Z_n \xrightarrow{P} c$ iff $Z_n \xrightarrow{D} c$ if c is a constant, and usual convergence $a_n \rightarrow a$ of a sequence of constants implies $a_n \xrightarrow{P} a$.

$$S_{X,Y} = \frac{1}{n-1} \hat{\sigma}_{X,Y} \xrightarrow{P} \sigma_{X,Y} \quad \text{by Slutsky's Th.}$$

$$\xrightarrow{P} \sigma_{X,Y} \quad \xrightarrow{D} \sigma_{X,Y}$$

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4) Suppose that the characteristic function of \bar{X}_n is

$$c_{\bar{X}}(t) = \exp\left(-\frac{t^2 \sigma^2}{2n}\right).$$

Then the characteristic function of $\sqrt{n} \bar{X}_n$ is $c_{\sqrt{n} \bar{X}_n}(t) = c_{\bar{X}}(\sqrt{n} t) = \exp\left(-\frac{t^2 \sigma^2}{2}\right)$. Does $\sqrt{n} \bar{X}_n \xrightarrow{D} W$ for some random variable W ? Explain.

$$\left(\begin{array}{l} (X_1, \dots, X_n \text{ iid } N(0, \sigma^2) \text{ so } \bar{X}_n \sim N(0, \frac{\sigma^2}{n}) \\ \text{so } \sqrt{n} \bar{X}_n \sim N(0, \sigma^2) \end{array} \right)$$

$$c_{\sqrt{n} \bar{X}_n}(t) = c_{\bar{X}_n}(\sqrt{n} t) = \exp\left(-\frac{t^2 \sigma^2}{2}\right) = c_W(t)$$

is continuous at 0

$$\therefore \bar{X}_n \xrightarrow{D} W$$

$$\text{(and } W \sim N(0, \sigma^2)\text{)}$$

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