$$\sigma_{ij} = \frac{\alpha_i(1-\alpha_j)}{f(\xi_{\alpha_i})\, f(\xi_{\alpha_j})}$$

$.1\,(1-.1) = \frac{9}{100}$

$.1\,(1-.9) = 1/100$

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$.9\,(1-.9) = 9/100$

1) Find the limiting distribution of

$$\sqrt{n}\left((\hat{\xi}_{n,0.9} - \hat{\xi}_{n,0.1}) - (\xi_{n,0.9} - \xi_{n,0.1})\right). = (*)$$

take home

Use Exam 2 review 66) and 67).

$$\sqrt{n}\left[\begin{pmatrix}\hat{\xi}_{.1}\\ \hat{\xi}_{.9}\end{pmatrix} - \begin{pmatrix}\xi_{.1}\\ \xi_{.9}\end{pmatrix}\right] \xrightarrow{D} N_2\left[\begin{pmatrix}0\\0\end{pmatrix}, \underbrace{\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}}_{\Sigma}\right]$$

$$(*) = (-1\ \ 1)\ LHS \xrightarrow{D} N(0, \sigma_A^2) \text{ where}$$

$$\sigma_A^2 = (-1\ \ 1)\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-\sigma_{11}+\sigma_{21} & -\sigma_{12}+\sigma_{22}\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$= \sigma_{11} - 2\sigma_{12} + \sigma_{22}, \quad \Sigma = \begin{pmatrix}\frac{9/100}{[f(\xi_{.1})]^2} & \frac{1/100}{f(\xi_{.1})f(\xi_{.9})}\\ & \frac{9/100}{[f(\xi_{.9})]^2}\end{pmatrix}$$

$$\text{so } \sigma_A^2 = \boxed{\frac{9}{100[f(\xi_{.1})]^2} - \frac{2}{100 f(\xi_{.1}) f(\xi_{.9})} + \frac{9}{100[f(\xi_{.9})]^2}}$$

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PM 2) Suppose that $\boldsymbol{\beta}$ is a $p \times 1$ vector and that $\sqrt{n}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \xrightarrow{D} N_p(\mathbf{0}, \boldsymbol{C})$ where $\boldsymbol{C}$ is a $p \times p$ nonsingular matrix. Let $\boldsymbol{A}$ be a $j \times p$ matrix with full rank j. Suppose that $\boldsymbol{A\beta} = \mathbf{0}$.

30 a) What is the limiting distribution of $\sqrt{n}\boldsymbol{A}\hat{\boldsymbol{\beta}}_n$? $= \sqrt{n}\, A(\hat{\beta}_n - \beta)$

$$\xrightarrow{D} N_j(\underset{\sim}{0}, ACA^T)$$

30 b) What is the limiting distribution of $\boldsymbol{Z}_n = \sqrt{n}[\boldsymbol{ACA}^T]^{-1/2}\boldsymbol{A}\hat{\boldsymbol{\beta}}_n$?

$\xrightarrow{D} N_j(\underset{\sim}{0}, I_j)$ since

$$[ACA^T]^{-\frac{1}{2}} ACA^T [ACA^T]^{-\frac{1}{2}} = I_j$$

20 c) What is the limiting distribution of $\boldsymbol{Z}_n^T \boldsymbol{Z}_n = n\hat{\boldsymbol{\beta}}_n^T \boldsymbol{A}^T[\boldsymbol{ACA}^T]^{-1}\boldsymbol{A}\hat{\boldsymbol{\beta}}_n$?
Hint: If $\boldsymbol{Z} \sim N_k(\mathbf{0}, \boldsymbol{I})$ the $\boldsymbol{Z}^T\boldsymbol{Z} \sim \chi^2_k$. ← use the hint given $Z_n \to Z \sim N_k(0, I)$ $\Rightarrow Z_n^T Z_n \to Z^T Z \sim \chi^2_k$

$$\xrightarrow{D} \underset{\sim}{Z}^T \underset{\sim}{Z} \sim \chi^2_j$$

(where $\underset{\sim}{Z} \sim N_j(0, I_j)$ and $\underset{\sim}{Z}^T \underset{\sim}{Z} \stackrel{D}{=} \underbrace{\sum_{i=1}^{j} Z_i^2}_{\chi^2_1}$ is a sum of iid $[N(0,1)]^2$ RVs.)

By b) $Z_n \xrightarrow{D} N_j(0, I_j)$ so by hint

$$Z_n^T Z_n \xrightarrow{D} \chi^2_j$$

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