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e 2) Let the linear model $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{X}$ has full rank p, $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$ and $Cov(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$. Then for a large class of iid error distributions, what is the limiting distribution of $\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$? Hint: use the least squares central limit theorem.

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{D} \boxed{N_p(\underline{0}, \sigma^2 W)}$$

$$\left(\text{where } \frac{X^T X}{n} \to W^{-1}\right)$$

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3) Suppose $\boldsymbol{Z}_n \xrightarrow{D} N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $\boldsymbol{A}$ be a constant $r \times k$ matrix. Find the limiting distribution of $\boldsymbol{A}(\boldsymbol{Z}_n - \boldsymbol{\mu})$.

$$\underline{Z}_n - \underline{\mu} \xrightarrow{D} N_k(\underline{0}, \Sigma)$$

$$A(\underline{Z}_n - \underline{\mu}) \xrightarrow{D} \boxed{N_r(\underline{0}, A\Sigma A^T)}$$

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3) Suppose $\boldsymbol{Y}^* \sim N_n(\boldsymbol{X}\hat{\boldsymbol{\beta}}, \sigma_n^2\boldsymbol{I}_n)$. Hence $Y_i^* = \boldsymbol{x}_i^T\hat{\boldsymbol{\beta}} + \epsilon_i^P$ where $E(\epsilon_i^P) = 0$ and $V(\epsilon_i^P) = \sigma_n^2$. Hence $\boldsymbol{AY}^* \sim N_g(\boldsymbol{AX}\hat{\boldsymbol{\beta}}, \sigma_n^2\boldsymbol{AA}^T)$ if $\boldsymbol{A}$ is a $g \times n$ constant matrix. Recall that $\boldsymbol{X}$ is an $n \times p$ constant matrix. Simplify quantities when possible.

20 a) What is the distribution of $\hat{\boldsymbol{\beta}}^* = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}^*$?

$$N_p\left((X^TX)^{-1}X^TX\hat{\beta},\ \sigma_n^2 (X^TX)^{-1}X^TX(X^TX)^{-1}\right)$$

$$\sim N_p\left[\hat{\beta},\ \sigma_n^2 (X^TX)^{-1}\right]$$

20 b) Using a), what is $E(\hat{\boldsymbol{\beta}}^*)$?

$$\boxed{\hat{\beta}}$$

$PX = X,\quad PX_I = X_I \quad X_I^T P = X_I$

10 c) Recall that $\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{PY}$. What is the distribution of $\hat{\boldsymbol{\beta}}_I^* = (\boldsymbol{X}_I^T\boldsymbol{X}_I)^{-1}\boldsymbol{X}_I^T\boldsymbol{Y}^*$?

$k \times 1$

$$N_k\left[\underbrace{(X_I^TX_I)^{-1}\underbrace{X_I^TP}_{X_I^T}Y}_{\hat{\beta}_I},\ \sigma_n^2 (X_I^TX_I)^{-1}X_I^T X_I(X_I^TX_I)^{-1}\right]$$

$$\boxed{\sim N_k\left[\hat{\beta}_I,\ \sigma_n^2 (X_I^TX_I)^{-1}\right]}$$

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