16} * If $F_{X_n}(t) \to F_X(t)$ at all continuity points of $F_X(t)$, then $X_n \xrightarrow{D} X$.

If t_0 is a discontinuity point of $F_X(t)$, the behavior of $F_{X_n}(t_0)$ is not important: could have $\lim_{n\to\infty} F_{X_n}(t_0) = c_{t_0} \in [0,1]$ or $\lim_{n\to\infty} F_{X_n}(t_0)$ does not exist.

Do not need $C_{t_0} = F_X(t_0)$.

17} If $F_{X_n}(t) \to H(t)$ except at discontinuity points of $F_X(t)$, still need $H(t) = F_X(t)$ at continuity points of $F_X(t)$.

18) Properties of a cdf for a RV:

i) $0 \le F(t) \le 1 \quad \forall t \in \mathbb{R}$

ii) $F(-\infty) = \lim_{t\to -\infty} F(t) = 0$

iii) $F(\infty) = \lim_{t\to\infty} F(t) = 1$

iv) $F(t)$ is nondecreasing:
$F(t_1) \le F(t_2)$ if $t_1 \le t_2$.

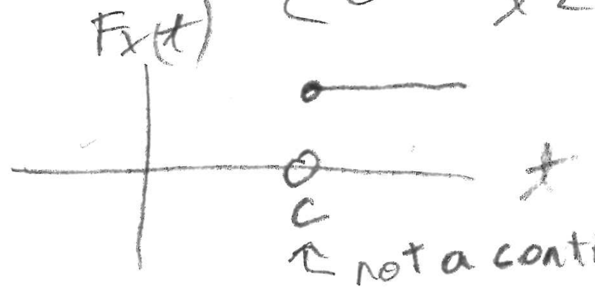
v) $F(t)$ is right continuous
$\lim_{t \downarrow t_0} F(t) = F(t_0)$ or $\lim_{h \downarrow 0} F(t+h) = F(t)$ $\forall t \in \mathbb{R}$.

vi) There are at most countably many discontinuity points of $F(t)$.

vii) If X is discrete with support $\{x_1, x_2, \ldots, x_k\}$ with $k = \infty$ possible,
then $F_X(t)$ is a step function with jumps at x_i and jump size $= P(X = x_i)$ at x_i. The x_i are the discontinuity points of $F_X(t)$.

19) X has a point mass distr at c or X is degenerate at c if $P(X = c) = 1$. So X has a pmf with $k = 1$ and $x_1 = c$.

Then $F_X(t) = \begin{cases} 1 & t \geq c \\ 0 & t < c. \end{cases}$



↑ not a continuity point of F_X

often $c = \mu$ or $c = \theta$

often $F_{X_n}(t) \to F_X(t) \quad \forall t \neq c$.

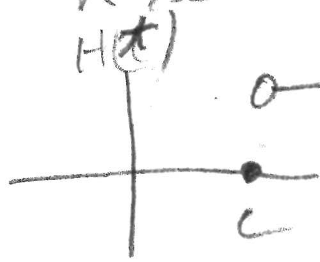
Then $X_n \xrightarrow{D} X$ where $P(X = c) = 1$.

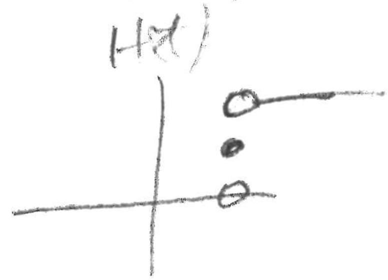
So $F_{X_n}(t) \to H(t) \quad \forall t \neq c$

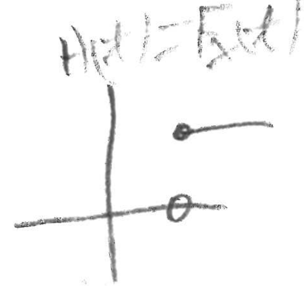
where $H(t) = F_X(t) \quad \forall t \neq c$.

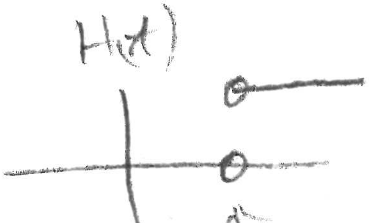
Could have $\lim_{n\to\infty} F_{X_n}(c) = H(c) \in [0,1]$

or $\lim_{n\to\infty} F_{X_n}(c)$ does not exist.









↑ $H(c)$ does not exist.

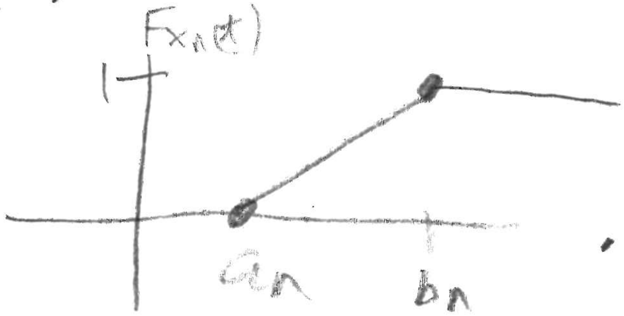
ex) Let $X_n \sim U(a_n, b_n)$ with $a_n < b_n$.

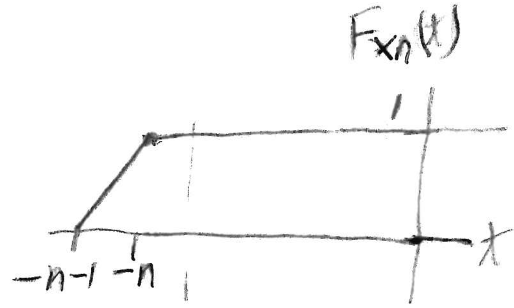
Then $F_{X_n}(t) = \dfrac{t - a_n}{b_n - a_n}$, $a_n \le t \le b_n$.

so $F_{X_n}(t) = \begin{cases} 0 & t \le a_n \\ 1 & t \ge b_n. \end{cases}$ } often omitted.

On $[a_n, b_n]$, $F_{X_n}(t)$ is a line segment

from $(a_n, 0)$ to $(b_n, 1)$ with slope $\frac{1}{b_n - a_n}$.





a) $X_n \sim U(-n-1, -n)$

$F_{X_n}(t) \to H(t) \equiv 1 \quad \forall t \in \mathbb{R}$.

$H(t)$ is continuous but not a cdf,

so X_n does not converge in

dist to any RV X.

b) $X_n \sim U(n, n+1)$

$F_{X_n}(t) \to H(t) \equiv 0 \quad \forall t \in \mathbb{R}$.

$H(t)$ is continuous but not a cdf.

So X_n does not converge in dist to any RV X.

c) $X_n \sim U(a_n, b_n)$, $a_n \to a < b$, $b_n \to b$.

Then
$$F_{X_n}(t) \to F_X(t) = \begin{cases} 0, & t \le a \\ \frac{t-a}{b-a}, & a \le t \le b \\ 1 & t \ge b \end{cases}$$

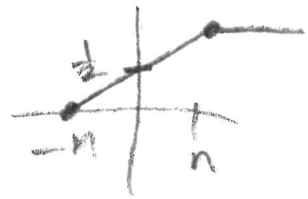
So $X_n \xrightarrow{D} X \sim U(a,b)$.

d) $X_n \sim U(a_n, b_n)$, $a_n \to c$, $b_n \to c$

Then
$$F_{X_n}(t) \to \begin{cases} 0 & t < c \\ 1 & t > c \\ ? & t = c \end{cases}$$

$\therefore$ $X_n \xrightarrow{D} X$ where $P(X=c) = 1$.

e) $X_n \sim U(-n, n)$



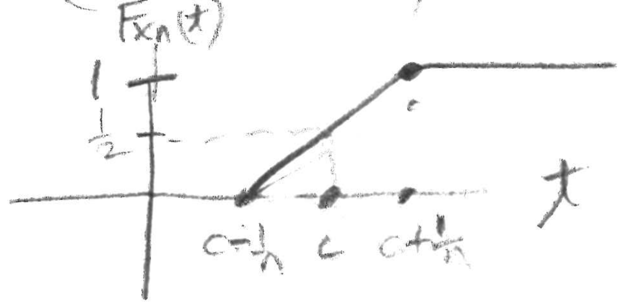
$$F_{X_n}(t) = \frac{t+n}{2n} = \frac{1}{2} + \frac{t}{2n}, \quad -n \le t \le n.$$

$$\lim_{n \to \infty} F_{X_n}(t) = H(t) = \frac{1}{2} \quad \forall t \in \mathbb{R}$$

$H(t)$ is continuous but not a cdf.

$\therefore$ X_n does not converge in dist to any RV X.

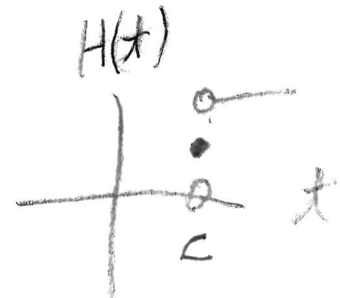
f) $X_n \sim U\left(c - \frac{1}{n}, c + \frac{1}{n}\right)$



$$F_{X_n}(t) = \frac{t - c + \frac{1}{n}}{\frac{2}{n}} = \frac{1}{2} + \frac{n}{2}(t-c),$$

for $c - \frac{1}{n} \le t \le c + \frac{1}{n}$.

$$F_{X_n}(t) \to H(t) = \begin{cases} 0 & t < c \\ \frac{1}{2} & t = c \\ 1 & t > c \end{cases}$$



$\therefore F_{X_n}(t) \to F_X(t) = \begin{cases} 0 & t<c \\ 1 & t \geq c \end{cases}$

except at $t=c$, the discontinuity point of $F_X(t)$.

$\therefore X_n \xrightarrow{D} X$ where $P(X=c)=1$.

common error: student says

$F_{X_n}(t) \to \begin{cases} \infty & t>c \\ -\infty & t<c \end{cases}$

but $0 \leq F_{X_n}(t) \leq 1 \quad \forall t$.

20) $X_n \xrightarrow{D} X$ iff $\lim_{n\to\infty} P[X_n \leq x] = P[X \leq x]$

for every $x \ni \underbrace{P[X=x]=0}_{\text{no jump in } F_X \text{ at } x}$

21) Write $Y \sim X$ or $Y \overset{D}{=} X$ if Y and X have the same dist (so same cdf and same characteristic function).

22} p48 know
convergence in probability 10.5

a) $X_n \xrightarrow{P} \underbrace{\tau(\theta)}_{\text{constant}}$ if for every $\varepsilon > 0$

$\lim_{n \to \infty} P(|X_n - \tau(\theta)| < \varepsilon) = 1$ or if

$\lim_{n \to \infty} P(|X_n - \tau(\theta)| \geq \varepsilon) = 0.$

b) $X_n \xrightarrow{P} X$ if for every $\varepsilon > 0$

$\lim_{n \to \infty} P(|X_n - X| < \varepsilon) = 1$ or if

$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0.$

23} know
Test statistic T_n is a
consistent estimator of $\tau(\theta)$
if $T_n \xrightarrow{P} \tau(\theta)$ $\forall\, \theta \in \Theta$.

24) Want any estimator T_n of $\tau(\theta)$ to be a consistent estimator of $\tau(\theta)$, T_n gets arb close to $\tau(\theta)$ with prob arb close to 1 as $n \to \infty$.

Inconsistent estimators are usually considered bad estimators.

25) * ~~Generalized Chebyshev's Inequality~~

= Generalized Markov's Inequality

Let $U: \mathbb{R} \to [0, \infty)$ be a nonnegative function and let $E[U(Y)]$ exist.

Then for any $c > 0$,

$$P[U(Y) \geq c] \leq \frac{E[U(Y)]}{c}.$$

Proof) $E[U(Y)] = \int_{\mathbb{R}} u(y) f(y)\, dy$

$$= \int_{\{y: u(y) \geq c\}} u(y) f(y)\, dy + \underbrace{\int_{\{y: u(y) < c\}} \overbrace{u(y) f(y)}^{\text{nonnegative}}\, dy}_{\geq 0}$$

$$\geq \int_{\{y \mid U(y) \geq c\}} U(y) f(y)\, dy \geq c \int_{\{y \,!\, U(y) \geq c\}} f(y)\, dy$$

$= c\, P[U(Y) \geq c]$.

Replace integrals by sums for pmfs.

$\square$

26) p48 * <u>Chebyshev's Inequality</u>

If $V(Y)$ exists, and $\mu = E(Y)$, then

$$P[|Y-\mu| \geq c] \leq \frac{V(Y)}{c^2} \text{ for any } c > 0.$$

proof! Take $U(y) = (y-\mu)^2$. Then

$$P[|Y-\mu| \geq c] = P[(Y-\mu)^2 \geq c^2] \leq \frac{V(Y)}{c^2}$$

by 25). $\square$

27) * <u>Markov's Inequality</u>:

Suppose $\mu = E(Y)$ and $E[|Y-\mu|^k]$ exist where $k > 0$. For any r with $0 < r \leq k$ and for any $c > 0$

$$P[|Y-\mu| \geq c] \leq \frac{E[|Y-\mu|^r]}{c^r}.$$

proof: If $E[|Y-\mu|^k]$ exists, then $E[|Y-\mu|^r]$ exists for $0 < r \leq k$.

Take $u(Y) = |Y-\mu|^r$ and $\tilde{c} = c^r$.

Then $P[|Y-\mu| \geq c] = P[|Y-\mu|^r \geq c^r]$

$$\leq \frac{E[|Y-\mu|^r]}{c^r} \text{ by } 25).\ \square$$

Note: 26) follows with $r = 2$ if $k \geq 2$.

28} want conditions

$A \Rightarrow B$ where $B = X_n \xrightarrow{P} X$.

If A does not hold, can't tell whether B holds.

<u>common error</u>} say A does not hold, so X_n does not converge to X in probability.

29} $MSE_{\tau(\theta)}(T_n) = E_\theta[(T_n - \tau(\theta))^2]$

$= V_\theta(T_n) + [bias_{\tau(\theta)}(T_n)]^2$

where $bias_{\tau(\theta)}(T_n) = E_\theta[T_n] - \tau(\theta)$.

30} a) If $\lim_{n\to\infty} MSE_{\tau(\theta)}(T_n) = \lim_{n\to\infty} E_\theta[(T_n - \tau(\theta))^2]$ see p49. *

$= 0 \ \forall \theta \in \Theta$, then T_n is a consistent estimator of $\tau(\theta)$.