such that

$$n^{\delta}(T_{1n} - \theta) \xrightarrow{D} N(0, \sigma_1^2(F))$$

and $n^{\delta}(T_{2n} - \theta) \xrightarrow{D} N(0, \sigma_2^2(F))$.

Then the <u>asymptotic relative efficiency</u> of T_{1n} wrt T_{2n} is

$$ARE(T_{1n}, T_{2n}) = \frac{\sigma_2^2(F)}{\sigma_1^2(F)}.$$

533 a) $0 < \delta \leq 1$ and usually $\delta = \frac{1}{2}$.

b) If T_{in} has n^{δ_i}, then T_{1n} is better than T_{2n} if $\delta_1 > \delta_2$.

c) Often 2 estimators T_{1n} and T_{2n} do not estimate the same parameter θ for many distributions F, unless the distributions are symmetric about θ.

c) For fixed F,

T_{1n} is better than T_{2n} if ARE > 1

worse ARE < 1.

So the better estimator has the smaller $\sigma_i^2(F)$.

ex} Let $Y_1, \ldots, Y_n$ be iid $N(\mu, \sigma^2)$.

Let $\theta = \mu = E(Y) = MED(Y)$.

Let $T_{1n} = \overline{Y}_n$ and $T_{2n} = MED(n)$.

By the CLT, $\sigma_1^2(F) = \sigma^2$.

By 49}, $\sigma_2^2(F) = \dfrac{1}{4[f(MED(Y))]^2}$

$$= \frac{1}{4\left[\frac{1}{\sqrt{2\pi\sigma^2}} \underbrace{\exp\left(\frac{-0}{2\sigma^2}\right)}_{1}\right]^2}$$

$$= \frac{2\pi\sigma^2}{4} = \frac{\pi\sigma^2}{2}.$$

$$ARE(T_{1n}, T_{2n}) = ARE(\overline{Y}_n, MED(n))$$

$$= \frac{(\pi\sigma^2/2)}{\sigma^2} = \frac{\pi}{2} \approx 1.5708$$

Hence $\overline{Y}_n$ is a better estimator of μ than $MED(n)$ for the family of $N(\mu, \sigma^2)$ distributions, $\sigma^2 > 0$.

54} p485-6 know Assume $\tau'(\theta) \neq 0$. 24.5

An estimator T_n of $\tau(\theta)$ is
asymptotically efficient if

$$\sqrt{n}\,(T_n - \tau(\theta)) \xrightarrow{D} N\left(0, \frac{[\tau'(\theta)]^2}{I_1(\theta)}\right)$$

$$\sim N(0, FCRLB_1(\tau(\theta))).$$

In particular, an estimator
T_n of θ is asymptotically efficient
if $\sqrt{n}\,(T_n - \theta) \xrightarrow{D} N\left(0, \frac{1}{I_1(\theta)}\right) \sim N[0, FCRLB_1(\theta)]$.

55} p486 If T_{2n} is an
asymptotically efficient estimator of θ,
if $I_1(\theta)$ and $v(\theta)$ are continuous
functions, and if $\sqrt{n}\,(T_{1n} - \theta) \xrightarrow{D} N(0, v(\theta))$,

Then under regularity conditions, $v(\theta) \geq \frac{1}{I_1(\theta)}$ and

so $ARE(T_{1n}, T_{2n}) = \frac{\left(\frac{1}{I_1(\theta)}\right)}{v(\theta)} =$

$\frac{1}{I_1(\theta)\, v(\theta)} \leq 1.$ Hence asymptotically efficient estimators are better than estimators of the form T_{1n}.

56) The standard limit theorem 40) says that under strong regularity conditions a) The MLE $\hat{\theta}_n$ of θ and the MLE $\tau(\hat{\theta}_n)$ of $\tau(\theta)$ are asymptotically efficient estimators of θ and $\tau(\theta)$.

b) $\tau(\hat{\theta}_n)$ is an asymptotically efficient estimator of $\tau(\theta)$

if $\hat{\theta}_n$ is the UMVUE of θ.

57) P70 T_n If $n^\delta(T_n - \tau(\theta)) \xrightarrow{D} X_\theta \quad \forall \theta \in \Theta$

then T_n is a consistent estimator of $\tau(\theta)$

where $0 < \delta \leq 1$,

Note: $X_\theta \sim N(0, v(\theta))$ and $\delta = 0.5$ are common.

If $\sqrt{n}(T_n - \theta) \xrightarrow{D} X_\theta$, then $T_n \xrightarrow{P} \theta$.

Know P70

58) Slutsky's Theorem: Suppose

$Y_n \xrightarrow{D} Y$ and $W_n \xrightarrow{P} c$ for

some constant c. Then

a) $Y_n + W_n \xrightarrow{D} Y + c$

b) $Y_n W_n \xrightarrow{D} cY$

c) $\dfrac{Y_n}{W_n} \xrightarrow{D} \dfrac{Y}{c}$ if $c \neq 0$.

Note: $w_n \xrightarrow{P} c$ iff $w_n \xrightarrow{D} c$.

regular convergence

If $c_n \to c$ then $c_n \xrightarrow{wp1} c$ and $c_n \xrightarrow{P} c$.

If $w_n \xrightarrow{A} c$, $A = wp1$ or r, then $w_n \xrightarrow{P} c$.

57) ps1-2 If $Y_1, \ldots, Y_n$ are iid with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$ then $S_n^2 \xrightarrow{P} \sigma^2$ and $S_M^2 \xrightarrow{P} \sigma^2$.

ex) a) $S_n^2 \xrightarrow{P} \sigma^2$ $\therefore$ $S_M^2 = \underbrace{\frac{n-1}{n}}_{\xrightarrow{P} 1} S_n^2 \xrightarrow{P} 1(\sigma^2) = \sigma^2$

by Slutsky's Th.

b) $S_M^2 \xrightarrow{P} \sigma^2$ $\therefore$ $S_n^2 = \frac{n}{n-1} S_M^2 \xrightarrow{P} (1)\sigma^2 = \sigma^2$.

c) If $Y_n \xrightarrow{A} d$ and $W_n \xrightarrow{B} c$ where c and d are constants and A or B are wp1, r, P or D, then $Y_n \xrightarrow{D} d$ and $W_n \xrightarrow{P} c$ so

$$Y_n + W_n \xrightarrow{P} d + c$$

$$Y_n W_n \xrightarrow{P} dc$$

$$\frac{Y_n}{W_n} \xrightarrow{P} \frac{d}{c} \quad \text{if } c \neq 0$$

In particular, if $Y_n \xrightarrow{D} d$ and $W_n \xrightarrow{D} c$ the above 3 results hold, and P can be replaced by D.

d) If $Y_n \sim t_n$, a t dist with n degrees of freedom, then $t_n \xrightarrow{D} Z \sim N(0,1)$

proof: $Y_n \stackrel{D}{=} \dfrac{Z}{\sqrt{V_n/n}}$ where $Z \perp\!\!\!\perp V_n \sim \chi^2_n$.

$V_n \stackrel{D}{=} \dfrac{\sum_{i=1}^{n} X_i}{n}$ where $X_i \stackrel{iid}{\sim} \chi^2_1$.

So $\frac{V_n}{n} \xrightarrow{P} 1 \quad \therefore \sqrt{\frac{V_n}{n}} \xrightarrow{P} \sqrt{1} = 1$ by 5&3

e) If $Y_n \sim F_{p,d_n}$, an F dist with p and d_n degrees of freedom, then

$p Y_n = p F_{p,d_n} \xrightarrow{D} \chi^2_p$ if $d_n \to \infty$ as $n \to \infty$.

proof $Y_n \sim F_{p,d_n} \overset{D}{=} \dfrac{X_1/p}{X_2/d_n}$ where

$X_1 \sim \chi^2_p \perp\!\!\!\perp X_2 \sim \chi^2_{d_n}$

$\therefore X_2 \overset{D}{=} \sum_{i=1}^{d_n} Y_i$, $Y_i \overset{iid}{\sim} \chi^2_1$ and

$\dfrac{X_2}{d_n} \xrightarrow{P} 1$ $\therefore p Y_n = \dfrac{1}{\frac{X_2}{d_n}} X_1 \xrightarrow{D} (1) X_1 \overset{D}{=} \chi^2_p$.

58) Let $g: \mathbb{R} \to \mathbb{R}$ be continuous at constant c.

a) If $X_n \xrightarrow{D} c$, then $g(X_n) \xrightarrow{D} g(c)$

p.51 b) If $X_n \xrightarrow{P} c$, then $g(X_n) \xrightarrow{P} g(c)$

c) If $X_n \xrightarrow{wp1} c$, then $g(X_n) \xrightarrow{wp1} g(c)$.

Note: If $X_n \xrightarrow{r} c$, then $X_n \xrightarrow{P} c$ so $g(X_n) \xrightarrow{P} g(c)$.

$E X_n \to \mu \not\Rightarrow E(X_n^2) \to (EX^2)$

59} <u>Generalized Continuous Mapping Th</u>

If $X_n \xrightarrow{D} X$ and the function g is such that $P[X \in C(g)] = 1$ where $C(g)$ is the set of points at which g is continuous, then $g(X_n) \xrightarrow{D} g(X)$.

Note! $P[X \in C(g)] = 1 \Leftrightarrow P[X \in D(g)] = 0$ where $D(g)$ is the set of points where g is not continuous.

ex} If $X_n \xrightarrow{D} X$ then $\frac{1}{X_n} \xrightarrow{D} \frac{1}{X}$

if X has a pdf (continuous RV) since $P(X=0) = 0$ and $x=0$ is the only discontinuity point of $g(x) = \frac{1}{x}$.

$\frac{1}{X_n} \xrightarrow{D} \frac{1}{X}$ if X has a pmf

with $P(X=0) = 0$.

60) <u>Continuous mapping Th</u>

If $X_n \xrightarrow{D} X$ and g is a continuous function, then $g(X_n) \xrightarrow{D} g(X)$.

Note) 60) is a special case of 59) where $D(g) = \phi$.

61) A complex RV $Z = X + iY$ where X and Y are ordinary RVs. Then $E(Z) = E(X) + i\,E(Y)$ where $i = \sqrt{-1}$. The modulus $|Z| = \sqrt{X^2 + Y^2}$. Much of the theory for Z parallels that of X.

In particular, linearity,

$|E(z)| \leq E[|z|]$, $z_1 \perp\!\!\!\perp z_2$

$\Rightarrow h_1(z_1) \perp\!\!\!\perp h_2(z_2)$ where h_i is a function of z_i alone.

often $z = e^{itX} = \cos(tX) + i\sin(tX)$.

62} Know The characteristic function

$C_X(t) = E\left(e^{itX}\right) = E[\cos(tX)] + iE[\sin(tX)]$.

real RV

63} $C_X(t)$ always exists and completely determines the dist of X.

i) $c(0) = 1$

ii) $|c(t)| \leq 1$ where the modulus

$|a+ib| = \sqrt{a^2+b^2}$.

iii) $c(t)$ is a continuous function.