64) If X is discrete with pmf $f(x)$, then $c_X(t) = \sum_x e^{itx} f(x)$.

If the mgf $m_X(t)$ exists, usually

$c_X(t) = m_X(it)$.

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65) <u>Continuity Th</u>

a) Let X_n be a sequence of RVs with char fn $c_{X_n}(t)$. Let X be a RV " " $c_X(t)$. a) Then

$X_n \xrightarrow{D} X$ iff $c_{X_n}(t) \to c_X(t)$ as $n \to \infty$ $\forall t \in \mathbb{R}$.

b) Also assume X_n has mgf $m_{X_n}(t)$ and X has mgf $m_X(t)$ where all mgfs are defined on $|t| \leq d$ for some $d > 0$. Then if

29.5

$M_{X_n}(t) \to M_X(t)$ as $n \to \infty$

for all $|t| < d$, (the value d need not be the largest possible value) then $X_n \xrightarrow{D} X$.

663 Th If $\lim_{n \to \infty} C_{X_n}(t) = g(t)$ $\forall t$

where g is continuous at $t=0$,

then $g = C_X(t)$ is a characteristic function and $X_n \xrightarrow{D} X$.

Warning: $C_{X_n}(0) \equiv 1$, but

$C_{X_n}(0) \to 1$ does not imply

g is continuous at $t=0$.

Note: Continuity at $t=0 \Rightarrow$ continuity everywhere since $g(t) = C_X(t)$ is continuous. If $g(t)$ is not continuous at $t=0$, then X_n does not converge in dist.

67} Let $X_1, \ldots, X_n$ be ind RVs with char functions $C_{X_j}(t)$.

Then $C_{\sum_{j=1}^{n} X_j}(t) = \prod_{j=1}^{n} C_{X_j}(t)$. If the RVs have mgfs $m_{X_j}(t)$, then

$$m_{\sum_{j=1}^{n} X_j}(t) = \prod_{j=1}^{n} m_{X_j}(t).$$

proof} $C_{\sum_{j=1}^{n} X_j}(t) = E\left[e^{it\sum_{j=1}^{n} X_j}\right]$

$$= E\left[e^{itX_1 + \cdots + itX_n}\right] = E\left[\prod_{j=1}^{n} e^{itX_j}\right]$$

$$\overset{\text{ind}}{=} \prod_{j=1}^{n} E\left[e^{itX_j}\right] = \prod_{j=1}^{n} C_{X_j}(t).$$

The proof for mgfs is the same except omit the i's and change c to m. $\square$

68) <u>Helly-Bray-Portmanteau</u> Th! 30.5

$X_n \xrightarrow{D} X$ iff $E[g(X_n)] \to E[g(X)]$

for every bounded, real, continuous function g.

proof of continuous mapping Th! If g is real and continuous, then $\cos[t\,g(x)]$ and $\sin[t\,g(x)]$ are bounded real continuous functions. Hence by the above th, for each $t \in \mathbb{R}$

acts as a constant
$t\,g(x)$ is contin

$$C_{g(X_n)}(t) = E[e^{it\,g(X_n)}] =$$

$$E(\cos[t\,g(X_n)]) + iE(\sin[t\,g(X_n)])$$

$$\to E(\cos[t\,g(X)]) + iE(\sin[t\,g(X)])$$

$$= E[e^{it\,g(X)}] = C_{g(X)}(t). \;\therefore$$

$g(X_n) \xrightarrow{D} g(X)$ by the continuity th.

$\square$

69) Notes for proving the CLT: LS

a) Let $\mathbb{C}$ be the set of complex numbers. Note $\mathbb{R} \subseteq \mathbb{C}$. Let $c_n \in \mathbb{R}$ or $\mathbb{C}$.

If $c_n \to c \in \mathbb{R}$, then

$$\lim_{n\to\infty} \left(1 - \frac{c_n}{n}\right)^n = e^{-c}.$$

b) Let $E(W) = 0$ and $V(W) = E(W^2) = \sigma^2$.

Then $c_W(t) = 1 - \frac{\sigma^2}{2} t^2 + o(t^2)$

is a Taylor series expansion of c_W where the remainder $R(t) = o(t^2)$

satisfies $\frac{o(t^2)}{t^2} \to 0$ as $t \to 0$.

c) $c_{aX}(t) = E\left[e^{itaX}\right] = c_X(at)$.

If $c_{X_n}(t) \to c_X(t)$ $\forall t \in \mathbb{R}$, then

at is a t^* and $c_{X_n}(at) = c_{aX_n}(t)$

$\to c_X(at) = c_{aX}(t)$ $\forall t$.

Hence if $Z_n = \frac{\sqrt{n}\,(\overline{Y}_n - \mu)}{\sigma} = \frac{\overline{Y}_n - \mu}{\sigma/\sqrt{n}}$

satisfies $C_{Z_n}(t) \to C_Z(t) = e^{-t^2/2}$, the

$N(0,1)$ char fn, then $\sigma Z_n = \sqrt{n}\,(\overline{Y}_n - \mu)$

has $C_{\sigma Z_n}(t) \to C_{\sigma Z}(t) = C_Z(\sigma t) = e^{-\sigma^2 t^2/2}$,

the $N(0, \sigma^2)$ char fn, and

$\sigma Z_n \xrightarrow{D} N(0, \sigma^2)$, so the CLT holds.

d) $Z_n = \frac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma\sqrt{n}} = \frac{\sum_{i=1}^{n} (Y_i - \mu)}{\sigma\sqrt{n}}$

where the $Y_i - \mu$ are iid with

char fn $C_{Y-\mu}(t)$. Hence the char fn

of $\frac{Y_i - \mu}{\sigma\sqrt{n}}$ is $C_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)$ and the

char fn of Z_n is $C_{Z_n}(t) \overset{\text{iid}}{=} \left[C_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$.

$E(Y_i - \mu) = 0$ and $V(Y_i-\mu) = E(Y_i-\mu)^2 = \sigma^2$.

So by c) $c_{Y-\mu}(t) = 1 - \frac{\sigma^2}{2} t^2 + o(t^2)$

and $c_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)$ (*)

where $\dfrac{o\left(\frac{t^2}{n}\right)}{\frac{t^2}{n}} \to 0$ as $n \to \infty$.

Hence $n\, o\left(\frac{t^2}{n}\right) \to 0$ as $n \to \infty$.

proof of CLT} By 69) $c_{Z_n}(t) =$

$$(*)^n = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n$$

$$= \left[1 - \frac{\frac{t^2}{2} - n\,o\left(\frac{t^2}{n}\right)}{n}\right]^n \to e^{-t^2/2}$$

$= c_Z(t)$ $\forall t$, $\therefore$ $Z_n \xrightarrow{D} Z \sim N(0,1)$

by the continuity th and $\sigma Z_n =$
$\sqrt{n}(\bar{Y}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$ by 69 c). $\square$

70) Recall $K(t) = \log[m(t)]$
is the cumulant generating function
with $K'(0) = E(X)$ and $K''(0) = V(X)$,
L'Hôspital's Rule: suppose $f(x) \to 0$ and $g(x) \to 0$
as $\underbrace{x \downarrow d,\ x \uparrow d,\ x \to d,\ x \to \infty \text{ or } x \to -\infty}$.

If $\frac{f'(x)}{g'(x)} \to L$, then $\frac{f(x)}{g(x)} \to L$ as ↑.

If the mgf exists, $E(X^k)$ exists $\forall k \in \mathbb{N}$.
(much stronger than σ^2 exists)

71) Proof of CLT <u>if mgf exists</u>!
Let $Y_1, Y_2, \ldots$ be iid with mean μ,
variance σ^2, and mgf $m_Y(t)$ for $|t| < t_0$.

Then $Z_i = \frac{Y_i - \mu}{\sigma}$ has mean 0, variance 1

and mgf $m_Z(t) = \exp\left(-\frac{t\mu}{\sigma}\right) m_Y\left(\frac{t}{\sigma}\right)$

for $|t| < \sigma t_0$. Want to show

$W_n = \sqrt{n}\left(\frac{\bar{Y}_n - \mu}{\sigma}\right) \xrightarrow{D} N(0,1).$

$W_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_i = n^{-\frac{1}{2}} \sum_{i=1}^{n} \left(\frac{Y_i - \mu}{\sigma}\right) =$

$n^{-\frac{1}{2}} \left(\frac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma}\right) = \frac{n^{-\frac{1}{2}}}{\frac{1}{n}} \; \frac{\bar{Y}_n - \mu}{\sigma}.$

$\therefore \; M_{W_n}(t) = E\left(e^{tW_n}\right) = E\left[\exp\left(t n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_i\right)\right]$

$= E\left[\exp\left(\sum_{i=1}^{n} \frac{tZ_i}{\sqrt{n}}\right)\right] \stackrel{ind}{=} \prod_{i=1}^{n} E\left[e^{tZ_i/\sqrt{n}}\right]$

$\stackrel{iid}{=} \prod_{i=1}^{n} M_Z\left(\frac{t}{\sqrt{n}}\right) = \left[M_Z\left(\frac{t}{\sqrt{n}}\right)\right]^n.$

Then $K_{W_n}(t) = n \log\left[M_Z\left(\frac{t}{\sqrt{n}}\right)\right] = n K_Z\left(\frac{t}{\sqrt{n}}\right)$

$= \frac{K_Z\left(\frac{t}{\sqrt{n}}\right)}{\frac{1}{n}}.$ Now $K_Z(0) = \log\left[M_Z(0)\right]$

$= \log(1) = 0.$

$\therefore$ by L'Hôspital's rule (deriv wrt $\frac{1}{n}$) 33.5

$$\lim_{n\to\infty} \log\left[M_{W_n}(t)\right] = \lim_{n\to\infty} \frac{K_Z\left(\frac{t}{\sqrt{n}}\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{K_Z'\left(\frac{t}{\sqrt{n}}\right)\left(\frac{-t}{2n^{3/2}}\right)}{-\frac{1}{n^2}} = \frac{t}{2} \lim_{n\to\infty} \frac{K_Z'\left(\frac{t}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}}.$$

Now $K_Z'(0) = \dfrac{M_Z'(0)}{M_Z(0)} = \dfrac{E(Z)}{1} = 0$,

so L'Hôspital's rule can be applied again:

$$\lim_{n\to\infty} \log\left[M_{W_n}(t)\right] = \frac{t}{2} \frac{K_Z''\left(\frac{t}{\sqrt{n}}\right)\left(\frac{-t}{2n^{3/2}}\right)}{\left(\frac{-1}{2n^{3/2}}\right)} =$$

$$\frac{t^2}{2} \lim_{n\to\infty} K_Z''\left(\frac{t}{\sqrt{n}}\right) = \frac{t^2}{2} \underbrace{K_Z''(0)}_{V(Z)=1} = \frac{t^2}{2}.$$

Thus $\lim_{n\to\infty} \log\left[M_{W_n}(t)\right] = \frac{t^2}{2}$

and $\lim_{n\to\infty} M_{W_n}(t) = e^{t^2/2}$, the $N(0,1)$ mgf.

$\therefore\ W_n = \sqrt{n}\left(\frac{\bar{Y}_n - \mu}{\sigma}\right) \xrightarrow{D} N(0,1)$ by

the continuity th. $\square$

ex 3 a) Let $X_n \sim bin(n, p_n)$ where $np_n = \lambda\ \forall n$.

Then the mgf $m_{X_n}(t) = (1 - p_n + p_n e^t)^n\ \forall t$.

$\therefore\ m_{X_n}(t) = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^t\right)^n =$

$$\left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n \to e^{\lambda(e^t-1)} = m_X(t)\ \forall t$$

where $X \sim POIS(\lambda)$.

$\therefore\ X_n \xrightarrow{D} X \sim POIS(\lambda)$ by

the continuity th

b) Now let $X_n \sim bin(n, p_n)$

where $np_n \to \lambda$ as $n \to \infty$.

$$M_{X_n}(t) = (1 - p_n + p_n e^t)^n \quad \forall t$$

$$= \left(1 + \frac{-np_n + np_n e^t}{n}\right)^n \to e^{\lambda(e^t - 1)} \quad \forall t$$

$\left(1 + \frac{c_n}{n}\right)^n \to e^c$ if $c_n \to c$

$c = -\lambda + \lambda e^t = \lambda(e^t - 1)$

and again $X_n \xrightarrow{D} X$.

Note: i) a) is easier and making assumptions that make large sample theory easier is a useful technique.

POIS(0) ⇒ point mass at 0

ii) For a given data set, $\hat{\lambda}_n = Y_n = n\hat{p}_n$ or $\hat{\lambda}_n = \max(1, Y_n)$

$X_n \sim bin(n, p_n) \approx POIS(\hat{\lambda})$ if $n\hat{p}_n = \hat{\lambda}$

where n is large and p_n small.