Math 584 HW 2 Spring 2021, due Thursday, Feb. 4.

1) Let $\boldsymbol{y}$ be an $n \times 1$ random vector with mean $\boldsymbol{\mu}$ and variance matrix $\text{Var}(\boldsymbol{y}) = \boldsymbol{\Sigma}$. Show that $E(\boldsymbol{y}\boldsymbol{y}') = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}'$ using the definition of the variance matrix (to prove $\boldsymbol{\Sigma} = E(\boldsymbol{y}\boldsymbol{y}') - \boldsymbol{\mu}\boldsymbol{\mu}'$).

2) Let $\boldsymbol{x}$ be an $n \times 1$ vector and let $\boldsymbol{B}$ be an $n \times n$ matrix. Show that $\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} = \boldsymbol{x}'\boldsymbol{B}'\boldsymbol{x}$. (The point of this problem is that if $\boldsymbol{B}$ is not a symmetric $n \times n$ matrix, then $\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ where $\boldsymbol{A} = \dfrac{\boldsymbol{B} + \boldsymbol{B}'}{2}$ is a symmetric $n \times n$ matrix.)

3) (Seber Ex.1a 2) on p. 9.) If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $m \times 1$ and $n \times 1$ vectors of random variables, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are $m \times 1$ and $n \times 1$ constant vectors, prove $Cov(\boldsymbol{X} - \boldsymbol{a}, \boldsymbol{Y} - \boldsymbol{b}) = Cov(\boldsymbol{X}, \boldsymbol{Y})$.

4) Let $Y \sim N(\mu, \sigma^2)$ so that $E(Y) = \mu$ and $\text{Var}(Y) = \sigma^2 = E(Y^2) - [E(Y)]^2$. If $k \geq 2$ is an integer, then

$$E(Y^k) = (k-1)\sigma^2 E(Y^{k-2}) + \mu E(Y^{k-1}).$$

Let $Z = (Y - \mu)/\sigma \sim N(0, 1)$. Hence $\mu_k = E(Y - \mu)^k = \sigma^k E(Z^k)$. Use this fact and the above recursion relationship $E(Z^k) = (k-1)E(Z^{k-2})$ to find a) μ_3 and b) μ_4.

5) Show that the matrix $\boldsymbol{A}$ in Example 1.9 ($A_{ij} = 1 - n^{-1}$ for $i = j$ and $A_{ij} = -n^{-1}$ for $i \neq j$, $\boldsymbol{A}$ is $n \times n$) satisfies

a) $\boldsymbol{A} = \boldsymbol{I} - \dfrac{1}{n}\boldsymbol{1}\boldsymbol{1}'$ where $\boldsymbol{1}$ is the $n \times 1$ vector of ones.

b) $\boldsymbol{A}^2 = \boldsymbol{A}$.

6) (Seber Ex.1b 5)ab on pp. 12-13.)
(Hint: see p. 10 and the two problems above. Also notice that $\boldsymbol{\theta} = \mu\boldsymbol{1}$. Show that $\boldsymbol{A}\boldsymbol{\theta} = \boldsymbol{0}$.)

Let $X_1, ..., X_n$ be iid $N(\mu, \sigma^2)$. Let $S^2 = \dfrac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2$ and $Q = \dfrac{1}{2(n-1)}\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2$.

a) Prove $Var(S^2) = 2\sigma^4/(n-1)$ using the following steps.

i) $V[(n-1)S^2] = V[\sum_{i=1}^n (x_i - \overline{x})^2] = V(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{X})$ using $\boldsymbol{A}$ from problem 5. Seber and Lee Theorem 1.6 says let $\boldsymbol{\theta} = E\boldsymbol{x} = \mu\boldsymbol{1}$ and $\boldsymbol{a} = (a_{11}, ..., a_{nn})'$. Then $V(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) = (\mu_4 - 3\mu_2^2)\boldsymbol{a}'\boldsymbol{a} + 2\mu_2^2 tr(\boldsymbol{A}^2) + 4\mu_2\boldsymbol{\theta}'\boldsymbol{A}^2\boldsymbol{\theta} + 4\mu_3\boldsymbol{\theta}'\boldsymbol{A}\boldsymbol{a}$. Using Problem 4, show $V(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) = 2\mu_2^2 tr(\boldsymbol{A}^2) + 4\mu_2\boldsymbol{\theta}'\boldsymbol{A}^2\boldsymbol{\theta} + 4\mu_3\boldsymbol{\theta}'\boldsymbol{A}\boldsymbol{a}$.

ii) Show $\boldsymbol{A}^2\boldsymbol{\theta} = \boldsymbol{A}\boldsymbol{\theta} = \boldsymbol{0}$.

iii) Then $V(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) = 2\mu_2^2 tr(\boldsymbol{A}^2) = 2\sigma^4 tr(\boldsymbol{A})$. Find $tr(\boldsymbol{A})$ then $V(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x})$.

iv) Find $V(S^2) = V(\frac{1}{n-1}(n-1)S^2) = V(\frac{1}{n-1}\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x})$.

b) Show Q is an unbiased estimate of σ^2 since $E[\sum_{i=1}^{n-1}(x_{i+1} - x_i)^2] = \sigma^2(2n-2)$.