Math 584 HW 3 Spring 2021, due Thursday, Feb. 11.
Problem 1 is useful for quiz 2.

For multiple linear regression review see ch.2 and ch. 3 of *Multiple Linear and 1D Regression* at (http://parker.ad.siu.edu/Olive/regbk.htm).

1) Let $\sigma_{12} = \text{Cov(Y,X)}$ and suppose Y and X follow a bivariate normal distribution

$$\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2\left(\begin{pmatrix} 100 \\ 49 \end{pmatrix}, \begin{pmatrix} 25 & \sigma_{12} \\ \sigma_{12} & 16 \end{pmatrix} \right).$$

a) If $\sigma_{12} = 0$, find the conditional distribution of $Y|X = 2$.

b) If $\sigma_{12} = 10$ find $E(Y|X)$ and $\text{Var}(Y|X)$.

c) If $\sigma_{12} = 10$, find $\rho(Y, X)$, the correlation between Y and X.

2) Let $X_1, ..., X_n$ be mutually independent random variables such that $X_i \sim N(\mu_i, \sigma_i^2)$. Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be fixed constants and let $W = \sum_{i=1}^n (a_i X_i + b_i)$. Use moment generating functions to find the distribution of W.

3) (Seber 2b 4 on p. 23 Modified:) Given $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{I}_n)$, find the joint distribution of

$$\begin{pmatrix} \boldsymbol{a}'\boldsymbol{Y} \\ \boldsymbol{b}'\boldsymbol{Y} \end{pmatrix}$$

where $\boldsymbol{a}^T\boldsymbol{b} = \boldsymbol{0}$. Here $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors.

4) (Seber 2b 7 on p. 24. Use the hint in the back of the book.) Let X_1 and X_2 have joint pdf

$$f(x_1, x_2) = \frac{1}{2\pi} \exp\left(\frac{-1}{2}(x_1^2 + x_2^2)\right) \left[1 - \frac{x_1 x_2}{(1 + x_1^2)(1 + x_2^2)}\right]$$

where $-\infty < x_1, x_2 < \infty$. Show $X_i \sim N(0, 1)$ for $i = 1, 2$.

Hint: Since the last term $g(x_i)$, say, is an odd function of x_i [i.e. $g(-x_i) = -g(x_i)$], this term vanishes when integrating over x_i.

5) (Seber 2c 1 on p. 26.) If $Y_1, ..., Y_n$ have a multivariate normal distribution and are pairwise independent, are they (mutually) independent?

Over for 6) and 7).

6) (Seber 2d 2a on p. 31. Use the hint in the back of the book and p. 17-18. Try to write the MGF as $MGF = c \int MVNpdf = c$.) Let $\boldsymbol{Y} \sim n_n(\boldsymbol{0}, \boldsymbol{I}_n)$. Show that the MGF of $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ is $[det(\boldsymbol{I}_n - 2t\boldsymbol{A})]^{-1/2}$.

Hint:

$$MGF = E[\exp(t\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y})] = \int e^{t\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}} \frac{1}{(2\pi)^{n/2}} \exp[\frac{-1}{2}(\boldsymbol{y}'\boldsymbol{I}_n\boldsymbol{y})]d\boldsymbol{y} =$$

$$\int \frac{1}{(2\pi)^{n/2}} \exp[\frac{-1}{2}\boldsymbol{y}'(\boldsymbol{I}_n - 2t\boldsymbol{A})\boldsymbol{y}]d\boldsymbol{y} = [det(\boldsymbol{I}_n - 2t\boldsymbol{A})]^{1/2}$$

for $|t|$ sufficiently small by A.4.9. Multiply the integral by

$$1 = \frac{|\boldsymbol{I}_n - 2t\boldsymbol{A}|^{-1/2}}{|\boldsymbol{I}_n - 2t\boldsymbol{A}|^{-1/2}}.$$

Should get MGF =

$$|\boldsymbol{I}_n - 2t\boldsymbol{A}|^{-1/2} \int N_n(\boldsymbol{0}, (\boldsymbol{I}_n - 2t\boldsymbol{A})^{-1})pdf.$$

7) (Seber 8 on p. 32.) Let $\boldsymbol{Y} \sim N_n(\boldsymbol{0}, \boldsymbol{I}_n)$ and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be symmetric idempotent matrices with $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A} = \boldsymbol{0}$. Show that $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}, \boldsymbol{Y}'\boldsymbol{B}\boldsymbol{Y}$ and $\boldsymbol{Y}'(\boldsymbol{I}_n - \boldsymbol{A} - \boldsymbol{B})\boldsymbol{Y}$ have independent chi-square distributions.

Hint: Theorem 2.7 shows that the quadratic forms are chi-square. Use Theorem 1.3 or Craig's theorem to show independence. That is, want to show $Cov(\boldsymbol{C}\boldsymbol{Y}, \boldsymbol{D}\boldsymbol{Y}) = \boldsymbol{C}Cov(\boldsymbol{Y})\boldsymbol{D}' = \boldsymbol{0}$ where $\boldsymbol{C}$ and $\boldsymbol{D}$ are the matrices in the center of the quadratic form. See Exam 1 review 12), 22).