Math 584   HW 4 Spring 2021, due Thursday, Feb. 26.
Exam 1 is Tu. Feb. 23 covering HW1-3, Q1-3.

1) Let $\boldsymbol{P}$ be the projection matrix on $C(\boldsymbol{X})$ so $\boldsymbol{X} = \boldsymbol{P}\boldsymbol{X}$. Let $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{Y}} = \boldsymbol{P}\boldsymbol{Y}$. Then $\hat{\boldsymbol{\theta}} \in C(\boldsymbol{X})$ and $\boldsymbol{P}\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}$. Show algebraically that $\boldsymbol{X}'(\boldsymbol{Y} - \hat{\boldsymbol{\theta}}) = 0$.

2) (Seber 3a 1 on p. 41.) Show that if $\boldsymbol{X}$ has full rank, then $(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}) = (\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) + (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\boldsymbol{X}'\boldsymbol{X}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$, and hence deduce that the left hand side is minimized uniquely when $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$.

Hint: Expand $(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}) = (\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}} + \boldsymbol{X}\hat{\boldsymbol{\beta}} - \boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}} + \boldsymbol{X}\hat{\boldsymbol{\beta}} - \boldsymbol{X}\boldsymbol{\beta})$, and use the result from problem 1).

3) (Seber 3a 2 on p. 41.) Assume that the first column of $\boldsymbol{X}$ is a vector of ones. If $\boldsymbol{X}$ has full rank, prove $\sum_{i=1}^n (Y_i - \hat{Y}_i) = 0$.

Hint: Consider the first column of $\boldsymbol{X}$.

4) (Seber 3a 6 on p. 41.) If $\boldsymbol{X}$ has full rank, so that $\boldsymbol{P} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$, prove that $C(\boldsymbol{P}) = C(\boldsymbol{X})$.

Hint: this problem is similar to a problem from Quiz 3.

5) (Seber 3a 7 on p. 42.) For a general regression model in which $\boldsymbol{X}$ may not have full rank, show that $\sum_{i=1}^n \hat{Y}_i(Y_i - \hat{Y}_i) = 0$.

Hint: $\hat{\boldsymbol{Y}} = \boldsymbol{P}\boldsymbol{Y}$ even if $\boldsymbol{X}$ is not full rank.