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Math 584    HW 8 Spring 2021, due Thursday, March 25.
Exam 2 is Tuesday, March 23.
Final: Tuesday, May 4, 2:45-4:45.
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1) The theory in chapters 4 and 5 is often used for experimental design, eg for pairwise comparisons and contrasts. Suppose that $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)'$. Consider testing both $\beta_1 = \beta_2$ and $\beta_1 = \beta_3$ simultaneously. Find $\boldsymbol{A}$ so that $H: \boldsymbol{A}\boldsymbol{\beta} = \boldsymbol{0}$ corresponds to this test.

2) Suppose that $\boldsymbol{X}$ is an $n \times p$ matrix but the rank of $\boldsymbol{X} < p < n$. Then the normal equations $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{X}'\boldsymbol{Y}$ have infinitely many solutions. Let $\hat{\boldsymbol{\beta}}$ be a solution to the normal equations. So $\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}'\boldsymbol{Y}$. Let $\boldsymbol{G} = (\boldsymbol{X}'\boldsymbol{X})^-$ be a generalized inverse of $(\boldsymbol{X}'\boldsymbol{X})$. Assume that $E(\boldsymbol{Y}) = \boldsymbol{X}\boldsymbol{\beta}$ and $\text{Cov}(\boldsymbol{Y}) = \sigma^2\boldsymbol{I}$. It can be shown that all solutions to the normal equations have the form $\boldsymbol{b_z}$ given below.

a) Show that $\boldsymbol{b_z} = \boldsymbol{G}\boldsymbol{X}'\boldsymbol{Y} + (\boldsymbol{G}\boldsymbol{X}'\boldsymbol{X} - \boldsymbol{I})\boldsymbol{z}$ is a solution to the normal equations where the $p \times 1$ vector $\boldsymbol{z}$ is arbitrary.

b) Show that $E(\boldsymbol{b_z}) \neq \boldsymbol{\beta}$.

(Hence some authors suggest that $\boldsymbol{b_z}$ should be called a solution to the normal equations but not an estimator of $\boldsymbol{\beta}$.)

c) Show that $\text{Cov}(\boldsymbol{b_z}) = \sigma^2\boldsymbol{G}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{G}'$.

d) Although $\boldsymbol{G}$ is not unique, the projection matrix $\boldsymbol{P} = \boldsymbol{X}\boldsymbol{G}\boldsymbol{X}'$ onto $\mathcal{C}(\boldsymbol{X})$ is unique. Use this fact to show that $\hat{\boldsymbol{Y}} = \boldsymbol{X}\boldsymbol{b_z}$ does not depend on $\boldsymbol{G}$ or $\boldsymbol{z}$.

e) From p. 64, there are two ways to show that $\boldsymbol{a}'\boldsymbol{\beta}$ is an estimable function. Either show that there exists a vector $\boldsymbol{c}$ such that $E(\boldsymbol{c}'\boldsymbol{Y}) = \boldsymbol{a}'\boldsymbol{\beta}$, or show that $\boldsymbol{a} \in \mathcal{C}(\boldsymbol{X}')$. Suppose that $\boldsymbol{a} = \boldsymbol{X}'\boldsymbol{w}$ for some fixed vector $\boldsymbol{w}$. Show that $E(\boldsymbol{a}'\boldsymbol{b_z}) = \boldsymbol{a}'\boldsymbol{\beta}$.

(Hence $\boldsymbol{a}'\boldsymbol{\beta}$ is estimable by $\boldsymbol{a}'\boldsymbol{b_z}$ where $\boldsymbol{b_z}$ is any solution of the normal equations.)

f) Suppose that $\boldsymbol{a} = \boldsymbol{X}'\boldsymbol{w}$ for some fixed vector $\boldsymbol{w}$. Show that $Var(\boldsymbol{a}'\boldsymbol{b_z}) = \sigma^2\boldsymbol{w}'\boldsymbol{P}\boldsymbol{w}$.

3) Let $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{Y} = (Y_1, Y_2, Y_3)'$, $\boldsymbol{\beta} = (\beta_1, \beta_2)'$, $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$, and $\text{Cov}(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$. If $\boldsymbol{X} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$, is $\boldsymbol{\beta}$ estimable? Explain briefly.

4) Let $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{Y} = (Y_1, Y_2, Y_3)'$, $\boldsymbol{X} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 2 & 6 \end{bmatrix}$, $\boldsymbol{\beta} = (\beta_1, \beta_2)'$, $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$, and $\text{Cov}(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$. Show whether or not the following functions are estimable.

a) $5\beta_1 + 15\beta_2$
b) β_1
c) $\beta_1 - 2\beta_2$