Spring 2019 Tu Th 50

75 minutes not 50 minutes

1.5 times longer

Math 584 2019 Exam 1 Name___________

Q2d21 e 1) Suppose that

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_4\left(\begin{pmatrix} 49 \\ 25 \\ 9 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 3 & 0 \\ -1 & 5 & -3 & 0 \\ 3 & -3 & 5 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \right).$$

a) Find the distribution of X_2.

$\boxed{\sim N(25, 5)}$

b) Find the distribution of $(X_1, X_3)^T$.

$\boxed{\sim N_2\left[\begin{pmatrix} 49 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\right]}$

c) Find the correlation $\rho(X_1, X_3)$. $= \frac{\mathrm{cov}(X_1, X_3)}{\sqrt{V(X_1)\, V(X_3)}} = \frac{3}{\sqrt{2}\sqrt{5}}$

$= \boxed{\frac{3}{\sqrt{10}} = 0.9487}$

21 e 2) Recall that if $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then the conditional distribution of $\boldsymbol{X}_1$ given that $\boldsymbol{X}_2 = \boldsymbol{x}_2$ is multivariate normal with mean $\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)$ and covariance matrix $\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$. Let Y and X follow a bivariate normal distribution

$$\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2\left(\begin{pmatrix} 49 \\ 17 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix} \right).$$

a) Find $E(Y|X)$. $= 49 + (-1)\frac{1}{4}(x-17) = 49 + \frac{17}{4} - \frac{1}{4}x$

$= \boxed{53.25 - 0.25x}$

b) Find $\mathrm{Var}(Y|X)$.

$= 3 - (-1)\frac{1}{4}(-1) = 3 - \frac{1}{4} = \boxed{2.75}$

14

3) Let $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\mathrm{E}(\boldsymbol{\epsilon}) = \boldsymbol{0}$ and $\mathrm{Cov}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{I}$. Assume $\boldsymbol{X}$ has full rank. Let $\boldsymbol{e}$ be the vector of residuals. Let $\boldsymbol{X}' = \boldsymbol{X}^T$.

a) Find $E(\boldsymbol{e})$.

$= E[(I-P)Y] = (I-P)X\beta = X\beta - X\beta = \boxed{0}$

b) Find $Cov(\boldsymbol{e})$. $= cov[(I-P)Y] = (I-P)\underbrace{cov(Y)}_{\sigma^2 I}(I-P)$

$= \boxed{\sigma^2 (I-P)}$

12

4) Let $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} > 0$. By Craig's theorem when is $\boldsymbol{Y}^T \boldsymbol{AY} \perp\!\!\!\perp \boldsymbol{Y}^T \boldsymbol{BY}$?

$\boxed{\text{if } A \Sigma B = 0}$ $(\Leftrightarrow B \Sigma A = 0)$

7

$(3\ \ 4)\begin{pmatrix}3\\4\end{pmatrix} = 9+16 = 25$

e 5) Find the projection matrix $\boldsymbol{P}$ for $\mathcal{C}(\boldsymbol{X})$ where $\boldsymbol{X}$ is the 2×1 vector $\boldsymbol{X} = (3, 4)'$.

$$= X(X^TX)^{-1}X^T = \begin{pmatrix}3\\4\end{pmatrix}\left[(3\ \ 4)\begin{pmatrix}3\\4\end{pmatrix}\right]^{-1}[3\ \ 4]$$

$$= \frac{1}{25}\begin{pmatrix}3\\4\end{pmatrix}(3\ \ 4) = \boxed{\frac{1}{25}\begin{bmatrix}9 & 12\\12 & 16\end{bmatrix} = \begin{bmatrix}0.36 & 0.48\\0.48 & 0.64\end{bmatrix}}$$

$\frac{}{7}$

6) Suppose the $n \times p$ matrix $\boldsymbol{X}$ has full rank $n < p$. Show that $\boldsymbol{X}^T(\boldsymbol{X}\boldsymbol{X}^T)^{-1}$ is a generalized inverse of $\boldsymbol{X}$.

Show $X X^- X = X$ or

$$\underbrace{X\ X^T(XX^T)^{-1}}_{I}\ X = X$$

$\frac{}{7}$

$Cov(\underline{Y}) = \sigma^2 I$, $A = I$, $E(\underline{Y}) = X\beta$

7 Let $\boldsymbol{Y} \sim N_n(\boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$. Recall that $E(\boldsymbol{Y'AY}) = tr(\boldsymbol{A}Cov(\boldsymbol{Y})) + E(\boldsymbol{Y'})\boldsymbol{A}E(\boldsymbol{Y})$. Find $E(\boldsymbol{Y'Y}) = E(\boldsymbol{Y'IY})$.

$= tr(I\sigma^2 I) + \beta' X' I X\beta$

$= \sigma^2 tr(I) + \beta' X' X\beta$

$= \boxed{\sigma^2 n + \underline{\beta}' X' X \underline{\beta}}$

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7 e Let $\boldsymbol{y} \sim N_2(\boldsymbol{\mu}, \sigma^2\boldsymbol{I})$ where $\boldsymbol{y} = (Y_1, Y_2)'$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)'$. Let $\boldsymbol{A} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix}$.

$A^2 = A = A'$
$B^2 = B = B'$

Are $\boldsymbol{Ay}$ and $\boldsymbol{By}$ independent? Explain.

$\boxed{\text{no } AB' \neq 0}$

no $AB \neq 0$ ok too

$\sigma^2 A B^T = 0$ ok

Th 2.5 $A\underline{Y} \perp\!\!\!\perp B\underline{Y}$ iff $A \Sigma B^T = 0$

iff $AB = 0$ since $\Sigma = Cov(\underline{Y}) = \sigma^2 I$

and $B = B^T$

7

$AB = \begin{bmatrix} \frac{1+\sqrt{3}}{8} & \frac{\sqrt{3}+3}{8} \\ \frac{1+\sqrt{3}}{8} & \frac{\sqrt{3}+3}{8} \end{bmatrix}$

9) Let $\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$.

$\left[\begin{matrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{matrix}\right], \left[\begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{matrix}\right]$ are also good problems

a) Find rank($\boldsymbol{X}$).

$\boxed{2}$

b) Find a basis for $\mathcal{C}(\boldsymbol{X})$.

$\boxed{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$

c) Find $[\mathcal{C}(\boldsymbol{X})]^{\perp}$ = nullspace of $\boldsymbol{X}^T$.

$X' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. $\underline{y} \in N(X')$ if

$y_1 + y_2 + y_3 = 0$ and $y_3 = 0$

or if $y_1 = -y_2$

$N(X') = [C(X')]^{\perp} = \boxed{\left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\}}$

$= sp\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = sp\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

18