10 in 50 pages (MWF 4 pages)

Math 584 2019 Exam 2 Name___________

2.15

1) Suppose $Y_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + \epsilon_i$ with $Q(\boldsymbol{\beta}) \geq 0$. Let c_n be a constant that does not depend on $\boldsymbol{\beta}$ or σ^2. Suppose the likelihood function is

WLS or OLS

$$L(\boldsymbol{\beta}, \sigma^2) = c_n \frac{1}{\sigma^n} \exp\left(\frac{-1}{2\sigma^2} Q(\boldsymbol{\beta})\right).$$

Q4d21

a) Suppose that $\hat{\boldsymbol{\beta}}_Q$ minimizes $Q(\boldsymbol{\beta})$. Show that $\hat{\boldsymbol{\beta}}_Q$ is the MLE of $\boldsymbol{\beta}$.

For fixed $\sigma > 0$, $L(\boldsymbol{\beta}, \sigma^2)$ is maximized by minimizing $Q(\boldsymbol{\beta}) \geq 0$. So $\hat{\boldsymbol{\beta}}_Q$ maximizes $L(\boldsymbol{\beta}, \sigma^2)$ regardless of the value of σ^2. So $\hat{\boldsymbol{\beta}}_Q$ is the MLE of $\boldsymbol{\beta}$.

b) Then find the MLE $\hat{\sigma}^2$ of σ^2. $= \tau$

B instead of $\hat{B}$, -1

$$\log L_p(\sigma^2) = d - \frac{n}{2}\log(\sigma^2) - \frac{Q}{2\sigma^2}$$

$$\frac{d \log L_p(\sigma^2)}{d\sigma^2} = \frac{-n}{2\sigma^2} + \frac{Q}{2\sigma^4} \stackrel{set}{=} 0 \quad \text{or} \quad \hat{\sigma}^2 = \frac{Q}{n} \quad \underline{\text{unique}}$$

or -2

$$\frac{d^2 \log L_p(\sigma^2)}{d(\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{2(\sigma^2)^{-3}}{2} Q \Big|_{\hat{\sigma}^2} = \frac{n}{2\hat{\sigma}^4} - \frac{n\hat{\sigma}^2}{\hat{\sigma}^6} = \frac{-n}{2\hat{\sigma}^4} < 0$$

$$= \frac{-n^3}{2Q^2}$$

or

$$\log L_p(\tau) = d - \frac{n}{2}\log(\tau) - \frac{Q}{2\tau}$$

$$\frac{d \log L_p(\tau)}{d\tau} = \frac{-n}{2\tau} + \frac{Q}{2\tau^2} \stackrel{set}{=} 0 \quad \text{or} \quad -n\tau + Q = 0 \quad \text{or} \quad \hat{\tau} = \hat{\sigma}^2 = \frac{Q}{n}$$

$\underline{\text{unique}}$

$$\frac{d^2 \log L_p(\tau)}{d\tau^2} = \frac{n}{2\tau^2} - \frac{2Q}{2\tau^3}\Big|_{\hat{\tau}} = \frac{n}{2\hat{\tau}^2} - \frac{2n\hat{\tau}}{2\hat{\tau}^3} = \frac{-n}{2\hat{\tau}^2} < 0$$

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So $\hat{\sigma}^2 = \frac{Q}{n}$ is the MLE $\quad Q = Q(\hat{\boldsymbol{\beta}})$

$\underline{a}'\beta$ is estimable if $\underline{a} \in C(X')$

if $E(b^T Y) = \underline{a}^T\beta$

full full rank $\underline{a}'\beta$ is estimable by $\underline{a}'\hat{\beta}$

2) Let $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$, $\text{Cov}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{I}_n$, and $\boldsymbol{X}$ has full rank. Note that $Y_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + \epsilon_i$.

Assume X is a constant matrix

a) Find $E(Y_i)$. $= \boxed{\underline{x}_i^T \underline{\beta}}$

$X = [\underline{x}_i'] = \begin{pmatrix} \underline{x}_1' \\ \vdots \\ \underline{x}_n' \end{pmatrix}$

Q6d21

$x_i \in$ rowspace(X)

so $X^T = [x_1, \ldots, x_n]$

3.14 $= \underline{x}_i'\underline{\beta} + \underbrace{E(\varepsilon_i)}_{0}$

b) Is $E(Y_i)$ estimable? Explain briefly. yes by $\underline{x}_i^T\hat{\underline{\beta}}$

or $\underline{x}_i \in C(X')$ since x_i is a column of X^T

or $\underline{b} = (0, \ldots, 0, \underset{i\text{th position}}{1}, 0, \ldots, 0)'$ since $\underline{b}^T \underline{Y} = Y_i$

or $\underline{x}_i^T\hat{\underline{\beta}} = \underline{x}_i^T (X'X)^{-1} X'\underline{Y} = \underline{d}'\underline{Y}$ with $E[\underline{d}'\underline{Y}] = E(Y_i)$

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3) Let $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon} \sim N_n(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_n)$. Assume $\boldsymbol{X}$ has full rank and that the first column of $\boldsymbol{X} = \boldsymbol{1}$ so that a constant is in the model. Let $\boldsymbol{e}$ be the vector of residuals. Then the residual sum of squares RSS $= \boldsymbol{e}'\boldsymbol{e} = \|(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{Y}\|^2$. The sample mean $\overline{\boldsymbol{Y}} = \frac{1}{n}\boldsymbol{1}^T\boldsymbol{Y}$. Prove that $\boldsymbol{e}'\boldsymbol{e}$ and $\overline{\boldsymbol{Y}}$ independent (or dependent).
(Hint: If $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $\boldsymbol{AY} \perp\!\!\!\perp \boldsymbol{BY}$ iff $\boldsymbol{A\Sigma B}^T = \boldsymbol{0}$.
So prove whether $(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{Y} \perp\!\!\!\perp \dfrac{1}{n}\boldsymbol{1}^T\boldsymbol{Y}$.)

$\underline{P}X = X$ so $\underline{P}\underline{1} = \underline{1}$

2.30

$(I-P)\ \sigma^2 I\ \frac{1}{n}\underline{1} = \frac{\sigma^2}{n}(\underline{1} - \underline{1}) = 0.$

So $\frac{1}{n}\underline{1}'\underline{Y} \perp\!\!\!\perp (I-P)\underline{Y}$.

(so $\bar{Y} \perp\!\!\!\perp \|(I-P)\underline{Y}\|^2$.)

not conformable

$\underset{n\times n}{I - \underline{P}}$ $\quad \underset{1\times n}{\cdot}$

or $\frac{1}{n}\underline{1}'\ \sigma^2 I\ (I-P) = \frac{\sigma^2}{n}(\underline{1}' - \underline{1}') = 0$

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$X^T\underline{P} = X^T$ so $\underline{1}'\underline{P} = \underline{1}'$

4) Partition $\boldsymbol{X}$ as $\boldsymbol{X} = [\boldsymbol{X}_1 \;\; \boldsymbol{X}_2]$, let $\boldsymbol{P}$ be the projection matrix for $C(\boldsymbol{X})$ and let $\boldsymbol{P}_1$ be the projection matrix for $C(\boldsymbol{X}_1)$. Find $\boldsymbol{P}\boldsymbol{P}_1$.

$$= \boxed{P_1}$$

$$C(P_1) = C(X_1) \subseteq C(X).$$

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5) Suppose that $Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i,4} + \epsilon_i$. Testing $H_0 : \beta_1 = \beta_3 = \beta_4 = 0$ is equivalent to testing $H_0 : \boldsymbol{A\beta} = \boldsymbol{0}$. What is $\boldsymbol{A}$?

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\text{want } A\beta = \begin{pmatrix} \beta_1 \\ \beta_3 \\ \beta_4 \end{pmatrix}.$$

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6) Suppose $\boldsymbol{Z}_n \xrightarrow{D} N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $\boldsymbol{A}$ be a constant $r \times k$ matrix. Find the limiting distribution of $\boldsymbol{AZ}_n$.

$$A Z_n \xrightarrow{D} \boxed{N_r\left(A\mu, A\Sigma A^T\right)}$$

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7) Let the linear model $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{X}$ has full rank p, $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$ and $\text{Cov}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{I}$. Let $\boldsymbol{A}$ be a conformable constant matrix. Then for a large class of iid error distributions, what is the limiting distribution of $\sqrt{n}\boldsymbol{A}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$? Let A be $r \times p$.

$$\sqrt{n}\, A(\hat{\beta} - \beta) \xrightarrow{D} \boxed{N_r\left(0, \sigma^2 A W A^T\right)}$$

$$\left(\text{by OLS CLT } \sqrt{n}(\hat{\beta} - \beta) \xrightarrow{D} N_p(0, \sigma^2 W) \text{ where } \frac{X^T X}{n} \to W^{-1}\right)$$

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8) Let the full model be $Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \epsilon_i$ and let the reduced model be $Y_i = \beta_0 + \beta_2 x_{i2} + \epsilon_i$ for $i = 1, ..., n$. Write the full model as $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\epsilon}$, and consider testing $H_0 : \boldsymbol{\beta}_2 = \boldsymbol{0}$ where $\boldsymbol{\beta}_1$ corresponds to the reduced model. Let $\boldsymbol{P}_1$ be the projection matrix on $C(\boldsymbol{X}_1)$ and let $\boldsymbol{P}$ be the projection matrix on $C(\boldsymbol{X})$.

Then $F_R = \frac{n-p}{q}\frac{\mathbf{Y}'(\boldsymbol{P}-\boldsymbol{P}_1)\mathbf{Y}}{\mathbf{Y}'(\boldsymbol{I}-\boldsymbol{P})\mathbf{Y}}$.

Q5d21

Assume $\boldsymbol{\epsilon} \sim N_n(\boldsymbol{0}, \sigma^2\boldsymbol{I})$. Assume H_0 is true.

a) What is q?

$= 6-2 = \boxed{4}$ = # predictors in full model but not in reduced model

2.31

b) What is the distribution of $\mathbf{Y}'(\boldsymbol{P}-\mathbf{P}_1)\boldsymbol{Y}$?

$\frac{\underline{Y}'(P-P_1)\underline{Y}}{\sigma^2} \sim \chi^2_q$ so $\underline{Y}'(P-P_1)\underline{Y} \sim \sigma^2\chi^2_q = \boxed{\sigma^2\chi^2_4}$

c) What is the distribution of $\mathbf{Y}'(\boldsymbol{I}-\mathbf{P})\boldsymbol{Y}$? rank $(I-P) = n-p = n-6$

$\frac{\underline{Y}'(I-P)\underline{Y}}{\sigma^2} \sim \chi^2_{n-p}$ so $\underline{Y}'(I-P)\underline{Y} \sim \sigma^2\chi^2_{n-p} \sim \boxed{\sigma^2\chi^2_{n-6}}$

d) What is the distribution of F_R?

$F_{q,n-p} \sim \boxed{F_{4,n-6}}$

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9) Let the model be $Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \epsilon_i$. The model in matrix form is $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon} \sim N_n(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$. Let $\boldsymbol{P}$ be the projection matrix on $C(\boldsymbol{X})$ where the $n \times p$ matrix $\boldsymbol{X}$ has full rank p. What is the distribution of $\boldsymbol{Y}'\boldsymbol{PY}$?

Hint: If $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{I})$, then $\boldsymbol{Y}^T\boldsymbol{AY} \sim \chi^2(\text{rank}(\boldsymbol{A}), \boldsymbol{\mu}^T\boldsymbol{A\mu}/2)$ iff $\boldsymbol{A} = \boldsymbol{A}'$ is idempotent. $\boldsymbol{Y} \sim N_n(\boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$, so $\dfrac{\boldsymbol{Y}}{\sigma} \sim N_n\left(\dfrac{\boldsymbol{X\beta}}{\sigma}, \boldsymbol{I}\right)$.

$$\frac{\underline{Y}'\underline{P}\underline{Y}}{\sigma^2} \sim \chi^2\left[\text{rank}(\underline{P}), \left(\frac{X\beta}{\sigma}\right)^T \frac{\underline{P}}{2} \frac{X\beta}{\sigma}\right] \sim \chi^2\left(\text{rank}(\underline{P}), \frac{(X\beta)'\underline{P}X\beta}{2\sigma^2}\right)$$

$$\text{So } \underline{Y}'\underline{P}\underline{Y} \sim \boxed{\sigma^2 \chi^2\left(\overset{=5}{p}, \frac{\beta' X' X\beta}{2\sigma^2}\right)}$$

$$\underline{P}X\beta = X\beta$$

$$\left(\begin{aligned} \text{rank}(\underline{P}) &= \text{tr}(\underline{P}) = \text{tr}\left(X(X'X)^{-1}X'\right) \\ &= \text{tr}\left[X'X(X'X)^{-1}\right] = \text{tr}(I_p) = p = \text{rank}(X) \end{aligned}\right)$$

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