Math 584 2019 Final Name___________

E2821 σ

2.14

1) Suppose $Y_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + \epsilon_i$ with $Q(\boldsymbol{\beta}) \geq 0$. Let c_n be a constant that does not depend on $\boldsymbol{\beta}$ or σ^2. Suppose the likelihood function is

$$L(\boldsymbol{\beta}, \sigma^{2}) = c_n \frac{1}{\sigma^n} \exp\left(\frac{-1}{\sigma} Q(\boldsymbol{\beta})\right).$$

a) Suppose that $\hat{\boldsymbol{\beta}}_Q$ minimizes $Q(\boldsymbol{\beta})$. Show that $\hat{\boldsymbol{\beta}}_Q$ is an MLE of $\boldsymbol{\beta}$.

For fixed $\sigma > 0$, $L(\beta, \sigma)$ is maximized by minimizing $Q(\beta)$.
so $\hat{\beta}_Q$ maximizes $L(\beta, \sigma)$ regardless of the value of $\sigma > 0$.
so $\hat{\beta}_Q$ is the MLE of β.

b) Then find an MLE $\hat{\sigma}$ of σ. $L_p(\sigma) = c_n \frac{1}{\sigma^n} \exp\left[\frac{-1}{\sigma} \underbrace{Q(\hat{\beta}_Q)}_{Q}\right]$

$\log L_p(\sigma) = \overset{\log c_n}{d_n} - n \log \sigma - \frac{1}{\sigma} Q$

$\frac{d \log L_p(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} Q \overset{set}{=} 0$ or $n\sigma = Q$

or $\hat{\sigma} = \frac{Q(\hat{\beta}_Q)}{n}$ unique

$\frac{d^2 \log L_p(\sigma)}{d\sigma^2} = \frac{n}{\sigma^2} - \frac{2}{\sigma^3} Q \Big|_{\hat{\sigma}} =$

$\frac{n}{\hat{\sigma}^2} - \frac{2n\hat{\sigma}}{\hat{\sigma}^3} = \frac{-n}{\hat{\sigma}^2} < 0$ so $\hat{\sigma}$ is the MLE.

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2) Let $\boldsymbol{P} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^-\boldsymbol{X}^T$ be the projection matrix onto the column space of $\boldsymbol{X}$. Using $\boldsymbol{P}\boldsymbol{X} = \boldsymbol{X}$, show $\boldsymbol{P}$ is idempotent.

$$PP = \underbrace{PX}_{X}(X^TX)^-X^T = P$$

3,2 $$\text{or } PP = \underbrace{X(X^TX)^-X^TX}_{PX=X}(X^TX)^-X^T = X(X^TX)^-X^T = P$$

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3) Show that $(\boldsymbol{X}\boldsymbol{X}^T)^{-1}\boldsymbol{X}$ is a generalized inverse of $\boldsymbol{X}^T$ if the $n \times p$ matrix $\boldsymbol{X}$ has $\text{rank}(\boldsymbol{X}) = n$.

$$X^T \underbrace{(XX^T)^{-1}XX^T}_{I} = X^T$$

$$AA^-A = A$$

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4) Write the following quantities as $\boldsymbol{a}'\boldsymbol{Y}$ or $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ or $\boldsymbol{A}\boldsymbol{Y}$.

a) e $\quad (I-P)Y$

→ b) $\Sigma_i(Y_i)^2$ $\quad Y'Y = Y'IY$

c) $\hat{\beta}$ $\quad (X'X)^{-1}X'Y$

d) $\Sigma_i Y_i$ $\quad 1'Y$

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5) Suppose that $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\text{Cov}(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2}$ where $\boldsymbol{\Sigma}^{1/2}$ is nonsingular and symmetric. Hence $\boldsymbol{\Sigma}^{-1/2}\boldsymbol{Y} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{X\beta} + \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}$. Find $\text{Cov}(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon})$. Simplify.

$= \Sigma^{-\frac{1}{2}} \text{cov}(\epsilon) \Sigma^{-\frac{1}{2}} = \sigma^2 \Sigma^{-\frac{1}{2}} \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} \Sigma^{-\frac{1}{2}}$

$= \boxed{\sigma^2 I}$

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6) Let $\boldsymbol{y} \sim N_2(\boldsymbol{\mu}, \sigma^2\boldsymbol{I})$ where $\boldsymbol{y} = (Y_1, Y_2)'$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)'$. Let $\boldsymbol{A} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$.

can multiply A by a and B by b.

Are $\boldsymbol{y}'\boldsymbol{Ay}$ and $\boldsymbol{y}'\boldsymbol{By}$ independent? Explain.

$AB = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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so yes by Craig's th

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7) Let $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{Y} = (Y_1, Y_2, Y_3)'$, $\boldsymbol{X} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\boldsymbol{\beta} = (\beta_1, \beta_2)'$, $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$,

and $\text{Cov}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{I}$.

a) Find $[\mathcal{C}(\boldsymbol{X}')]$. $\quad X' = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ so $\mathcal{C}(X') = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

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Show whether or not the following functions are estimable.

b) $5\beta_1 + 10\beta_2$ $= (5 \;\; 10)\underset{\sim}{\beta}$

$\begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in C(X')$

yes $\boxed{\text{estimable}}$

c) β_1 $= (1 \;\; 0)\underset{\sim}{\beta}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin C(X')$

no, $\boxed{\text{not estimable}}$

d) $\beta_1 - 2\beta_2$ $= (1 \;\; -2)\underset{\sim}{\beta}$, $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \notin C(X')$

no $\boxed{\text{not estimable}}$

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8) Assuming the assumptions of the least squares central limit theorem hold, what is the limiting distribution of $\sqrt{n}\ (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ if $(\boldsymbol{X}'\boldsymbol{X})/n \to \boldsymbol{W}^{-1}$ as $n \to \infty$?

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$$\sqrt{n}\ (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{D} \boxed{N_p(\underset{\sim}{0}, \sigma^2 W)}$$

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E2021

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9) Let the model be $Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \ldots + \beta_9 x_{i9} + \epsilon_i$. The model in matrix form is $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon} \sim N_n(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$. Let $\boldsymbol{P}$ be the projection matrix on $C(\boldsymbol{X})$ where the $n \times p$ matrix $\boldsymbol{X}$ has full rank p. What is the distribution of $\boldsymbol{Y}'\boldsymbol{PY}$?

Hint: If $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{I})$, then $\boldsymbol{Y}^T \boldsymbol{AY} \sim \chi^2(\text{rank}(\boldsymbol{A}), \boldsymbol{\mu}^T \boldsymbol{A\mu}/2)$ iff $\boldsymbol{A} = \boldsymbol{A}'$ is idempotent. $\boldsymbol{Y} \sim N_n(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{I})$, so $\dfrac{\boldsymbol{Y}}{\sigma} \sim N_n\left(\dfrac{\boldsymbol{X\beta}}{\sigma}, \boldsymbol{I}\right)$. Simplify.

$$\frac{Y'PY}{\sigma^2} \sim \chi^2\left(\text{rank}(P), \left(\frac{X\beta}{\sigma}\right)^T \frac{P}{2} \frac{X\beta}{\sigma}\right)$$

$$\sim \chi^2\left(\text{rank}(P), \frac{(X\beta)' P X\beta}{2\sigma^2}\right)$$

$$\text{so } \underset{\sim}{Y}' P \underset{\sim}{Y} \sim \boxed{\sigma^2 \chi^2\left(p = 10, \frac{\beta' X' X \beta}{2\sigma^2}\right)}$$

$$P X \beta = X\beta$$

$$\text{rank}(P) = tr(P) = tr\left(X (X'X)^{-1} X'\right)$$

$$= tr\left[X'X (X'X)^{-1}\right] = tr(I_p) = p = \text{rank}(X)$$

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big constraint

diagonal Σ_ϵ

10) To bootstrap the multivariate linear regression model with the parametric bootstrap, let $\boldsymbol{Y}_j^* \sim N_n(\boldsymbol{X}\hat{\boldsymbol{\beta}}_j, \hat{\sigma}_j^2 \boldsymbol{I}_n)$ for $j = 1, ..., m$. Then $\hat{\boldsymbol{\beta}}_j^* = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}_j^*$ for $j = 1, ..., m$, and $\hat{\boldsymbol{B}}^* = [\hat{\boldsymbol{\beta}}_1^*, \hat{\boldsymbol{\beta}}_2^*, ..., \hat{\boldsymbol{\beta}}_m^*]$. Simplify quantities when possible. **Show work!**

a) What is the distribution of $\hat{\boldsymbol{\beta}}_j^* = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}_j^*$?

$$N_p\left[(X'X)^{-1}X' X\hat{\beta}_j, \ \hat{\sigma}_j^2 (X'X)^{-1} X' I X (X'X)^{-1}\right]$$

$$\sim \boxed{N_p\left[\hat{\beta}_j, \ \hat{\sigma}_j^2 (X'X)^{-1}\right]}$$

b) Using a), what is $E(\hat{\boldsymbol{\beta}}_j^*)$?

$$\boxed{\hat{\beta}_j}$$

c) Recall that $\boldsymbol{X}\hat{\boldsymbol{\beta}}_j = \boldsymbol{P}\boldsymbol{Y}_j$. What is the distribution of $\hat{\boldsymbol{\beta}}_{jI}^* = (\boldsymbol{X}_I^T\boldsymbol{X}_I)^{-1}\boldsymbol{X}_I^T\boldsymbol{Y}_j^*$ if $\hat{\boldsymbol{\beta}}_{jI}^*$ is $k \times 1$? Note that $\hat{\beta}_{jI} = (X_I^T X_I)^{-1} X_I^T Y_j$.

$$N_k\left[(X_I' X_I)^{-1} \underbrace{X_I' \overset{X\hat{\beta}_j}{P Y_j}}_{X_I'}, \ \hat{\sigma}_j^2 (X_I' X_I)^{-1} X_I' I X_I (X_I' X_I)^{-1}\right]$$

$$\sim \boxed{N_k\left[\hat{\beta}_{jI}, \ \hat{\sigma}_j^2 (X_I' X_I)^{-1}\right]}$$

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