Math 584 Quiz 3 Spring 2019 Name____________________

1) Let $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\mathrm{E}(\boldsymbol{\epsilon}) = \boldsymbol{0}$ and $\mathrm{Cov}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{I}$. Assume $\boldsymbol{X}$ has full rank. Let $\boldsymbol{e}$ be the vector of residuals. Let $\boldsymbol{X}' = \boldsymbol{X}^T$. **Show work, not just the final answer.**

a) Find $E(\boldsymbol{Y})$. $= X\beta + E(\epsilon) = \boxed{X\beta}$

b) Find $E(\hat{\boldsymbol{Y}})$. $= E X(X'X)^{-1}X'Y = X\underbrace{(X'X)^{-1}X'X}_{I}\beta = \boxed{X\beta}$

$= E(PY) = P\,E(Y) = PX\beta = X\beta$

$= E(X\hat{\beta}) = X\,E(\hat{\beta}) = X\beta$

c) Find $E(\boldsymbol{e})$. $= E[(I-P)Y] = (I-P)X\beta = \boxed{0}$

$= EY - E\hat{Y} = X\beta - X\beta = 0$

d) Show $\boldsymbol{e}'\hat{\boldsymbol{Y}} = 0$. $= [(I-P)Y]'PY = Y'(I-P)PY =$

$Y'(P-P)Y = \boxed{0}$

e) Find $\mathrm{Cov}(\hat{\boldsymbol{Y}})$. $\mathrm{cov}(PY) = P\,\mathrm{cov}(Y)P = P\sigma^2 IP = \boxed{\sigma^2 P}$

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$\mathrm{Cov}(Y) = \mathrm{Cov}(\epsilon) = \sigma^2 I$

$= \mathrm{cov}(X\hat{\beta}) = X\,\mathrm{cov}(\hat{\beta})X' = X\sigma^2(X'X)^{-1}X' = \sigma^2 P$

$X'(X'X)^{-1}X'X = X$ is wrong not +

2) When $\boldsymbol{X}$ is not full rank, the projection matrix $\boldsymbol{P}_X$ for $\mathcal{C}(\boldsymbol{X})$ is $\boldsymbol{P}_X = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$ by B.1.8.

Since $\boldsymbol{P}_X$ is the projection matrix for $\mathcal{C}(\boldsymbol{X})$, what is $\boldsymbol{P}_X\boldsymbol{X}$?

$\boxed{X}$

memorize $PX = X$
if P_X is the proj matrix for $C(X)$

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3) Let $\boldsymbol{X}$ be full rank and let $\boldsymbol{P} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$. To show that $\mathcal{C}(\boldsymbol{P}) = \mathcal{C}(\boldsymbol{X})$, can show that i) $\boldsymbol{Pw} = \boldsymbol{Xy} \in C(\boldsymbol{X})$ where $\boldsymbol{w}$ is an arbitrary conformable constant vector, and ii) $\boldsymbol{Xy} = \boldsymbol{Pw} \in C(\boldsymbol{P})$ where $\boldsymbol{y}$ is an arbitrary conformable constant vector.

i) Show $\boldsymbol{Pw} = \boldsymbol{Xy}$ and identify $\boldsymbol{y}$.

$P\underline{w} = X\underbrace{(X'X)^{-1}X'\underline{w}}_{\underline{y}} = X\underline{y}$ with $\underline{y} = (X'X)^{-1}X'\underline{w}$.

ii) Show $\boldsymbol{Xy} = \boldsymbol{Pw}$ and identify $\boldsymbol{w}$.

$X\underline{y} \underset{\underline{x}=P\underline{x}}{=} PX\underline{y} = P\underline{w}$ with $\underline{w} = X\underline{y}$

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