Math 584 Quiz 4 Spring 2019 Name____________________

1) Suppose $Y_i = \boldsymbol{x}_i^T\boldsymbol{\beta}+\epsilon_i$ where the errors are independent $N(0, \sigma^2/W_i)$ where $W_i > 0$ are known constants. Then the likelihood function is

$$L(\boldsymbol{\beta}, \sigma^2) = \left(\prod_{i=1}^n \sqrt{W_i}\right)\left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sigma^n}\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^n W_i(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2\right).$$

a) Suppose that $\hat{\boldsymbol{\beta}}_W$ minimizes $\sum_{i=1}^n W_i(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2$. By direct maximization, show that $\hat{\boldsymbol{\beta}}_W$ is the MLE of $\boldsymbol{\beta}$ regardless of the value of σ.

For fixed $\sigma > 0$, $L(\beta, \sigma^2)$ is maximized by minimizing $\sum W_i (y_i - x_i'\beta)^2 \geq 0$. So $\hat{\beta}_W$ maximized $L(\beta, \sigma)$ regardless of the value of σ^2.

So $\hat{\beta}_W$ is the MLE of β.

b) Then find the MLE $\hat{\sigma}^2$ of σ^2. $\tau = \sigma^2$

$$\log L_p(\sigma^2) = c - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\underbrace{\sum W_i (y_i - x_i\hat{\beta}_W)^2}_{Q}$$

$$\log L_p(\tau) = c - \frac{n}{2}\log(\tau) - \frac{Q}{2\tau} \qquad \left(\log L_p(\sigma^2) = c - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}Q\right)$$

$$\frac{d \log L_p(\tau)}{d\tau} = \frac{-n}{2\tau} + \frac{Q}{2\tau^2} \overset{set}{=} 0 \qquad \left(\frac{d}{d\sigma^2}\log L_p(\sigma^2) = \frac{-n}{2\sigma^2} + \frac{Q}{2\sigma^4}\right)$$

$$\frac{-n}{\tau} + Q = 0 \text{ or } n\tau = Q \qquad \text{or } \hat{\tau} = \frac{Q}{n} = \hat{\sigma}^2 \quad \underline{\text{unique}}$$

$$\frac{d^2 \log L_p(\tau)}{d\tau^2} = \frac{n}{2\tau^2} - \frac{2Q}{2\tau^3}\Big|_{\hat{\tau}} = \frac{n}{2\hat{\tau}^2} - \frac{2n\hat{\tau}}{2\hat{\tau}^3} = \frac{-n}{2\hat{\tau}^2} < 0$$

So $\hat{\sigma}^2$ is the MLE

$$\left(\frac{d^2 \log L_p(\sigma^2)}{d(\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{2(\sigma^2)^{-3}Q}{2}\Big|_{\sigma^2=\hat{\sigma}^2} = \frac{n}{2\hat{\sigma}^4} - \frac{n\hat{\sigma}^2}{\hat{\sigma}^6} = \frac{-n}{2\hat{\sigma}^4} < 0\right)$$

2) Let the linear model $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{X}$ has full rank p, $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$ and $Cov(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$. Then for a large class of iid error distributions, what is the limiting distribution of $\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$? Hint: use the least squares central limit theorem.

$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{D} \boxed{N_p(0, \sigma^2 W)}$

(where $\frac{X^TX}{n} \to W^{-1}$)

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3) Suppose $\boldsymbol{Z}_n \xrightarrow{D} N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $\boldsymbol{A}$ be a constant $r \times k$ matrix. Find the limiting distribution of $\boldsymbol{A}(\boldsymbol{Z}_n - \boldsymbol{\mu})$.

$Z_n - \mu \xrightarrow{D} N_k(0, \Sigma)$

$A(Z_n - \mu) \xrightarrow{D} \boxed{N_r(0, A\Sigma A')}$

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