Math 595 Quiz 3 Spring 2021 Name____________________

1) Let $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ where $\mathrm{E}(\boldsymbol{\epsilon}) = \boldsymbol{0}$ and $\mathrm{Cov}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{I}$. Assume $\boldsymbol{X}$ has full rank. Let $\boldsymbol{e}$ be the vector of residuals. Let $\boldsymbol{X}' = \boldsymbol{X}^T$.

a) Show $\boldsymbol{X}'\boldsymbol{e} = \boldsymbol{0}$.

b) Find $E(\boldsymbol{X}'\boldsymbol{e})$.

c) Find $E(\boldsymbol{e})$.

d) Show $\boldsymbol{e}'\hat{\boldsymbol{Y}} = 0$.

e) Show $\mathrm{Cov}(\boldsymbol{e}, \hat{\boldsymbol{Y}}) = \boldsymbol{0}$.

2) When $\boldsymbol{X}$ is not full rank, the projection matrix $\boldsymbol{P}_X$ for $\mathcal{C}(\boldsymbol{X})$ is $\boldsymbol{P}_X = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$ by B.1.8. To show that $\mathcal{C}(\boldsymbol{P}_X) = \mathcal{C}(\boldsymbol{X})$, can show that i) $\boldsymbol{P}_X\boldsymbol{w} = \boldsymbol{X}\boldsymbol{y} \in C(\boldsymbol{X})$ where $\boldsymbol{w}$ is an arbitrary conformable constant vector, and ii) $\boldsymbol{X}\boldsymbol{y} = \boldsymbol{P}_X\boldsymbol{w} \in C(\boldsymbol{P}_X)$ where $\boldsymbol{y}$ is an arbitrary conformable constant vector.

i) Show $\boldsymbol{P}_X\boldsymbol{w} = \boldsymbol{X}\boldsymbol{y}$ and identify $\boldsymbol{y}$.

ii) Show $\boldsymbol{X}\boldsymbol{y} = \boldsymbol{P}_X\boldsymbol{w}$ and identify $\boldsymbol{w}$. Hint: $\boldsymbol{P}_X\boldsymbol{X} = \boldsymbol{X}$.

iii) Since $\boldsymbol{P}_X$ is the projection matrix for $\mathcal{C}(\boldsymbol{X})$, what is $\boldsymbol{X}^T\boldsymbol{P}_X$?