$$E\left[\underset{\sim}{X}^T A \underset{\sim}{X}\right] = \operatorname{tr}(A\Sigma) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}.$$

proof} (Also see proof in text.)

$$E\left(\underset{\sim}{X}^T A \underset{\sim}{X}\right) = E\left[\operatorname{tr}\left(\underset{\sim}{X}^T A \underset{\sim}{X}\right)\right]$$

tr(scalar) = scalar

tr(AB) = tr(BA)

$(E(\Sigma X_{ii}) = \Sigma E(X_{ii}) = E[\operatorname{tr}(X)] = \operatorname{tr}[E(X)]$.

$$= E\left[\operatorname{tr}\left(\underset{\sim}{X}\,\underset{\sim}{X}^T A\right)\right] = \operatorname{tr}\left[E\left(\underset{\sim}{X}\,\underset{\sim}{X}^T A\right)\right]$$

$$= \operatorname{tr}\left[E\left(\underset{\sim}{X}\,\underset{\sim}{X}^T\right) A\right] = \operatorname{tr}\left[\left(\Sigma + \underset{\sim}{\mu}\,\underset{\sim}{\mu}^T\right) A\right]$$

$\Sigma = E(\underset{\sim}{X}\,\underset{\sim}{X}^T) - \underset{\sim}{\mu}\,\underset{\sim}{\mu}^T$ so $E\underset{\sim}{X}\,\underset{\sim}{X}^T = \Sigma + \underset{\sim}{\mu}\,\underset{\sim}{\mu}^T$

$$= \operatorname{tr}\left(\Sigma A + \underset{\sim}{\mu}\,\underset{\sim}{\mu}^T A\right) = \operatorname{tr}(\Sigma A) + \operatorname{tr}\left(\underset{\sim}{\mu}\,\underset{\sim}{\mu}^T A\right)$$

$\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$

$$= \operatorname{tr}(\Sigma A) + \operatorname{tr}(\underbrace{\underset{\sim}{\mu}^T A \underset{\sim}{\mu}}_{\text{scalar}}) = \operatorname{tr}(\Sigma A) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}$$

$\square$

13} P13 The moment generating function of a RV $\underset{\sim}{X}$

is $M_{\underset{\sim}{X}}(\underset{\sim}{t}) = E\left[\exp\left(\underset{\sim}{t}^T \underset{\sim}{X}\right)\right]$ if the expectation exists $\forall \ \|\underset{\sim}{t}\| \le a_0$ for some $a_0 > 0$.

9.5

14) p13 [h] Let $\underset{\sim}{Y} = \begin{pmatrix} \underset{\sim}{Y_1} \\ \underset{\sim}{Y_2} \end{pmatrix}$ and $\underset{\sim}{t} = \begin{pmatrix} \underset{\sim}{t_1} \\ \underset{\sim}{t_2} \end{pmatrix}$.

If $m_{\underset{\sim}{Y}}(\underset{\sim}{t})$ exists, then $\underset{\sim}{Y_1}$ and $\underset{\sim}{Y_2}$ are independent iff

$$m_{\underset{\sim}{Y}}(\underset{\sim}{t}) = m_{\underset{\sim}{Y_1}}(\underset{\sim}{t_1}) \ m_{\underset{\sim}{Y_2}}(\underset{\sim}{t_2})$$

where $m_{\underset{\sim}{Y_1}}(\underset{\sim}{t_1}) = m_{\underset{\sim}{Y}}\begin{pmatrix} \underset{\sim}{t_1} \\ \underset{\sim}{0} \end{pmatrix}$ and $m_{\underset{\sim}{Y_2}}(\underset{\sim}{t_2}) = m_{\underset{\sim}{Y}}\begin{pmatrix} \underset{\sim}{0} \\ \underset{\sim}{t_2} \end{pmatrix}$

Note: often write $m_{\underset{\sim}{Y}}(\underset{\sim}{t}) = m_{\underset{\sim}{Y}}(t_1, \ldots, t_n)$ row vector

where $m_{Y}(\underset{\sim}{t}) = E[\exp(\underset{\sim}{t}^T \underset{\sim}{Y})]$

$\underset{\sim}{t}^T$ is a row vector

see p17-20,22

Ch 2) 1) A $p \times 1$ random vector $\underset{\sim}{X}$ has a **p-dimensional multivariate normal (MVN) distribution**

$\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$ iff $\underset{\sim}{t}^T \underset{\sim}{X}$ has a univariate normal distribution for any $p \times 1$ constant vector $\underset{\sim}{t}$. $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$, $Cov(\underset{\sim}{X}) = \Sigma$, $m_{\underset{\sim}{X}}(\underset{\sim}{t}) = \exp(\underset{\sim}{t}^T \underset{\sim}{\mu} + \frac{1}{2} \underset{\sim}{t}^T \Sigma$

2) "Usually want Σ positive definite so $\underline{X}$ has pdf $f(\underline{z}) =$

$$\frac{1}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\underline{z}-\underline{\mu})^T \Sigma^{-1}(\underline{z}-\underline{\mu})\right] \text{ where}$$

$|\Sigma| = \det(\Sigma)$.

ex) If $p=1$, $\Sigma = \sigma^2$, $(z-\mu)^T \Sigma^{-1}(z-\mu) = \frac{(z-\mu)^2}{\sigma^2}$

3) If $\underline{X} = (X_1, \ldots, X_p)^T$ where the X_i are ind $N(\mu_i, \sigma_i^2)$, then $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ where $\underline{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$ and $\Sigma = \text{diag}(\sigma_i^2)$.

4) p20 *If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ and A is a $q \times p$ constant matrix, then $A\underline{X} \sim N_q(A\underline{\mu}, A\Sigma A^T)$.

If $\underline{a}$ $p\times1$ and $\underline{b}$ $q\times1$ are constant vectors, then

$\underline{a} + \underline{X} \sim N_p(\underline{\mu}+\underline{a}, \Sigma)$, and

$A\underline{X}+\underline{b} \sim N_q(A\underline{\mu}+\underline{b}, A\Sigma A^T)$

5) * If $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$, then all subsets are MVN i.e. $(X_{k_1}, \ldots, X_{k_q})^T \sim N_q(\tilde{\mu}, \tilde{\Sigma})$

where $\tilde{\mu}_i = E X_{k_i}$ and $\tilde{\Sigma}_{ij} = \operatorname{Cov}(X_{k_i}, X_{k_j})$.

ex) know for E1, Q2

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left[\begin{pmatrix} 1 \\ 17 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right]$$

Find dist of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ and $\begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$.

Soln $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 1 \\ 17 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\right]$

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\right]$$

$$\begin{pmatrix} X_2 \\ X_3 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 17 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right]$$

6) Let $\underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X_1} \\ \underset{\sim}{X_2} \end{pmatrix} \begin{matrix} q \times 1 \\ \\ (p-q) \times 1 \end{matrix}$, $\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

7) know for E1 Q2 If $\underset{\sim}{X} \sim N_p(\underline{\mu}, \Sigma)$ then

$\underset{\sim}{X}_1 \sim N_q(\underline{\mu}_1, \Sigma_{11})$ and $\underline{X}_2 \sim N_{p-q}(\underline{\mu}_2, \Sigma_{22})$

8) p24 know for E1 Q2 If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$,

then $\underline{X}_1$ and $\underline{X}_2$ are independent

$(\underline{X}_1 \perp\!\!\!\perp \underline{X}_2)$ iff $\Sigma_{12} = \underset{\sim}{0}$ ← matrix of 0s.

9) However if the joint distribution of

$\underline{X}_1$ and $\underline{X}_2$ is not MVN, $\underline{X}_1$ and $\underline{X}_2$

could be uncorrelated but not independent

10) know for E1 Q2 The pop correlation

between 2 random variables X and Y is

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{V(X)\,V(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \rho(Y,X)$$

if $\sigma_X > 0$ and $\sigma_Y > 0$. If $\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2(\underline{\mu}, \Sigma)$,

then $\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2\left[\begin{pmatrix} \mu_Y \\ \mu_X \end{pmatrix}, \begin{pmatrix} \sigma_Y^2 & \sigma_{YX} \\ \sigma_{XY} & \sigma_X^2 \end{pmatrix}\right]$

where $\sigma_{YX} = \text{cov}(Y,X) = \text{cov}(X,Y) = \sigma_{XY}$.

ex} $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3 \left[\begin{pmatrix} 1 \\ 17 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right]$.

a) which variables are ind? $X_1 \perp\!\!\!\perp X_3$ and $X_2 \perp\!\!\!\perp X_3$

b) $\rho(X_1, X_2) = \dfrac{1}{\sqrt{4}\sqrt{3}} = 0.2887$

c) $\rho(X_1, X_3) = 0$

d) $\rho(X_2, X_3) = 0$

ex] know for E1 $\underset{p \times 1}{\underline{Y}} \sim N_p(\underline{\mu}, \sigma^2 I)$.

Find the dist of $\underset{q \times p}{A}\, \underset{\sim}{Y}$.

Soln $A\underset{\sim}{Y} \sim N_q(A\underline{\mu}, A\sigma^2 I A^T)$

$\sim N_q(A\underset{\sim}{\mu}, \sigma^2 A A^T)$.

11} know for E1 (p25-6) If $\underset{\sim}{X} \sim N_p(\underline{\mu}, \Sigma)$ then the conditional distribution

$$\underline{X}_1 \mid \underline{X}_2 = \underline{x}_2 \sim N_q\Big[\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2),\ \underbrace{\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}_{\text{free of } \underline{X}_2 = \underline{x}_2}\Big]$$

Notation} $$\underset{\sim}{X}_1 \mid \underset{\sim}{X}_2 \sim N_q\Big[\underbrace{\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}_2 - \underline{\mu}_2)}_{\text{family of means depending on } X_2},\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Big]$$

ex) $\begin{pmatrix} Y \\ X_1 \\ \vdots \\ X_p \end{pmatrix} \sim N_{p+1}\left[\begin{pmatrix} E(Y) \\ E(\underline{X}) \end{pmatrix}, \begin{pmatrix} \overset{1\times 1}{V(Y)} & \overset{1\times p}{\Sigma_{Y\underline{X}}} \\ \Sigma_{\underline{X}Y} & \underset{p\times p}{\Sigma_{\underline{X}\underline{X}}} \end{pmatrix}\right]$

where $\Sigma_{\underline{X}\underline{X}} = Cov(\underline{X})$ and $\Sigma_{Y\underline{X}} = E[\underbrace{(Y-EY)}_{\text{scalar}}(\underline{X}-E\underline{X})^T]$

Then $E(Y|\underline{X}=\underline{x}) = EY + \Sigma_{Y\underline{X}}\Sigma_{\underline{X}\underline{X}}^{-1}(\underline{x}-E\underline{X})$

$\alpha = EY - \beta^T E\underline{X}$

$= EY - \underbrace{\Sigma_{Y\underline{X}}\Sigma_{\underline{X}\underline{X}}^{-1}}_{\beta^T} E\underline{X} + \Sigma_{Y\underline{X}}\Sigma_{\underline{X}\underline{X}}^{-1}\underline{x} = \alpha + \underline{\beta}^T\underline{x}$

and $V(Y|\underline{X}=\underline{x}) = V(Y) - \Sigma_{Y\underline{X}}\Sigma_{\underline{X}\underline{X}}^{-1}\Sigma_{\underline{X}Y} \equiv \sigma^2,$

a constant free of $\underline{X}$. same formulas hold if $\binom{Y}{\underline{X}}$ are iid with

If the joint dist of $\begin{pmatrix} Y \\ \underline{X} \end{pmatrix} \sim N_{p+1}(\underline{\mu}, \Sigma)$

then $Y|\underline{X}=\underline{x}$ follows a multiple linear regression model: $Y|X=x = \alpha + \underline{\beta}^T\underline{x} + \varepsilon$, $\varepsilon \sim N(0,\sigma^2)$.

ex) bivariate normal $\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2\left[\begin{pmatrix} EY \\ EX \end{pmatrix}, \begin{pmatrix} V(Y) & cov(X,Y) \\ cov(X,Y) & V(X) \end{pmatrix}\right]$

since $cov(X,Y) = cov(Y,X)$. Then

$Y|X=x \sim N_1[E(Y|X=x), V(Y|X=x)]$ where

$E(Y|X=x) = EY + cov(X,Y)\frac{1}{V(X)}(x-EX) =$ $\rho(X,Y) = \frac{cov}{\sqrt{V(X)V}}$

12.5

$$EY + \rho(X,Y)\sqrt{\frac{V(Y)}{V(X)}}\,(x - EX) = \text{[illegible]}$$

$$EY - \rho(X,Y)\sqrt{\frac{V(Y)}{V(X)}}\,EX + \rho(X,Y)\sqrt{\frac{V(Y)}{V(X)}}\,x = \alpha + \beta x,$$

So the mean function is a line that passes through $(E(X), E(Y))$.

$$V(Y \mid X = x) = V(Y) - \operatorname{cov}(X,Y)\frac{1}{V(X)}\operatorname{cov}(X,Y)$$

$$= V(Y) - \rho(X,Y)\sqrt{\frac{V(Y)}{V(X)}}\,\rho(X,Y)\sqrt{V(X)\,V(Y)}$$

$$= V(Y) - [\rho(X,Y)]^2\,V(Y) = V(Y)\left(1 - [\rho(X,Y)]^2\right).$$

p25 Th 2.5: This holds for $\Sigma \geq 0$. Use $\underset{\sim}{Z} = \begin{pmatrix} \underset{\sim}{U} \\ \underset{\sim}{W} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \underset{\sim}{Y}$

12] * Let $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \Sigma)$, $\underset{\sim}{U} = A\underset{\sim}{Y}$, $\underset{\sim}{W} = B\underset{\sim}{Y}$

$A\underset{\sim}{Y} \perp\!\!\!\perp B\underset{\sim}{Y}$ iff $\operatorname{cov}(\underset{\sim}{U}, \underset{\sim}{W}) = \boxed{A \Sigma B^T = 0 \text{ iff } B \Sigma A^T = 0}$ equivalent by transpose

ex] $\Sigma = \sigma^2 I_n$ $\quad A\underline{Y} \perp\!\!\!\perp B\underline{Y}$ iff $AB^T = 0$ iff $BA^T = 0$

ex] $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \sigma^2 I_n)$

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \underbrace{\frac{1}{n}\underset{\sim}{1}^T}_{A}\underset{\sim}{Y}, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

$n \times 1$ vector of 1s

$$\begin{pmatrix} Y_1 - \bar{Y} \\ \vdots \\ Y_n - \bar{Y} \end{pmatrix} = (I_n - n^{-1} J_n)\,\underline{Y} = B\,\underline{Y} \quad \text{where } J_n = \underset{\sim}{1}\underset{\sim}{1}^T$$

Then $\quad A \Sigma B^T \overset{B = B^T}{=} \frac{1}{n}\underset{\sim}{1}^T \sigma^2 I_n \left(I_n - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)$

$$= \sigma^2 \left[\frac{1}{n} \underset{\sim}{1}^T - \frac{1}{n^2} \underbrace{\underset{-}{1}^T \underset{-}{1}}_{n} \, 1^T \right] = \underset{\sim}{0}$$

So $\bar{Y} \perp\!\!\!\perp (Y_1 - \bar{Y}, \dots, Y_n - \bar{Y})^T$. $\therefore$ $\bar{Y} \perp\!\!\!\perp S^2$.

13] know for E1 Q2 be able to use 12]

§2.4 Quadratic form theory when $\underset{\sim}{x} \sim N_n(\mu, \Sigma)$

14] p27 Recall that for quadratic form

$\underset{\sim}{x}^T A \underset{\sim}{x}$, A is symmetric.

15] p28 Th2.7 Let $\underset{\sim}{Y} \sim N_n(\underset{\sim}{0}, I_n)$ and let

A be symmetric.

$\underset{\sim}{Y}^T A \underset{\sim}{Y} \sim \chi^2_r$ iff A is idempotent

of rank $r = tr(A)$ since then A is a projection matrix. (symmetric and idempotent)

ex] $\underset{\sim}{Y}^T \underset{\sim}{Y} = \underset{\sim}{Y}^T I \underset{\sim}{Y} = \sum_{i=1}^n Z_i^2 \sim \chi^2_n$

where $Z_i \sim N(0,1)$.

16] p29 Th. Let $\underset{\sim}{Y} \sim N_n(\underset{\sim}{0}, I_n)$ with A, B symmetric.

If $\underset{\sim}{Y}^T A \underset{\sim}{Y} \sim \chi^2_r$ and $\underset{\sim}{Y}^T B \underset{\sim}{Y} \sim \chi^2_d$

Then $\underline{Y}'A\underline{Y} \perp\!\!\!\perp \underline{Y}'B\underline{Y}$ iff $AB = 0$.

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Note! $\underline{Y}^T C \underline{Y} = \underline{Y}^T C^T C \underline{Y} = \|C\underline{Y}\|^2 \sim \chi^2_k$ since

C is symmetric and idempotent by Th 2.7.

So $\underline{Y}^T A \underline{Y} \overset{=\|A\underline{Y}\|^2}{=}$ is a function of $A\underline{Y}$ and $\underline{Y}^T B \underline{Y}^{=\|B\underline{Y}\|}$

is a function of $B\underline{Y}$. Hence the theorem holds by Th 2.5 of 12}.

17} p30 (Th 2.8 and corollary) $\underset{\sim}{Y} \sim N_n(\underline{0}, \Sigma)$, $A = A^T$, $\underline{\Sigma > 0}$

Then $\underline{Y}^T A \underline{Y} \sim \chi^2_r$ iff r of the eigenvalues of $A\Sigma$ are 1 and the rest are 0

So $\underline{Y}^T A \underline{Y} \sim \chi^2_r$ iff $A\Sigma$ is idempotent of rank r. $(A\Sigma)^T = \Sigma^T A^T = \Sigma A$

18}✓ Th 2.9 If $Y \sim N_n(\underline{\mu}, \Sigma)$, $\Sigma > 0$,

then $\underbrace{(\underline{Y} - \underline{\mu})^T \Sigma^{-1} (\underline{Y} - \underline{\mu})}_{\text{pop squared Mahalanobis distance}} \sim \chi^2_n$

proof $\underline{Y} - \underset{\sim}{\mu} \sim N_n(\underline{0}, \Sigma)$ and $\underset{\sim}{Z} = \Sigma^{-\frac{1}{2}}(\underline{Y} - \underline{\mu}) \sim N_n(\underline{0}, I_n)$

So $(\underline{Y} - \underline{\mu})^T \Sigma^{-1} (\underline{Y} - \underline{\mu}) = \sum_{i=1}^{n} Z_i^2 \sim \chi^2_n$.

19) Let $Y_1, ..., Y_n$ be ind $N(\mu_i, 1)$ RVs so $\underset{\sim}{Y} = (Y_1, ..., Y_n)^T \sim N_n(\underset{\sim}{\mu}, I)$

Then $\underset{\sim}{Y}^T \underset{\sim}{Y} = \sum_{i=1}^n Y_i^2 \sim \chi^2(n, \frac{\underset{\sim}{\mu}^T \underset{\sim}{\mu}}{2})$,

a noncentral χ^2 distribution with degrees of freedom n and noncentrality parameter $\gamma = \frac{\underset{\sim}{\mu}^T \underset{\sim}{\mu}}{2} = \frac{1}{2} \sum_{i=1}^n \mu_i^2 \geq 0$. Or $Y^T Y \sim \chi^2(n, \delta)$ where $\delta = \mu^T\mu$. Note $\gamma = \frac{\delta}{2}$, $\delta = 2\gamma$.

20) If $W \sim \chi^2_n$ then $W \sim \chi^2(n, 0)$,

so $\gamma = 0$. The χ^2_n distribution is also called a central χ^2 distribution.

see chrisensen p8 LM reg cdf 1.14 M is singular unless M = I

21) If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{0}, M)$ where $M = M^T = M^2$ (symmetric idempotent projection matrix),

then $\underset{\sim}{Y}^T \underset{\sim}{Y} \sim \chi^2_{tr(M)}$ where $tr(M) = rank(M)$

22) a) If $Y \sim \chi^2(n, \gamma)$ then the mgf of Y is $m_Y(t) = (1-2t)^{-n/2} e^{-\gamma[1-(1-2t)^{-1}]}$ for $t < \frac{1}{2}$.
$= (1-2t)^{-n/2} \exp[2\gamma t (1-2t)^{-1}]$

b) If $Y_i \sim \chi^2(n_i, \gamma_i)$ are ind for $i = 1, \ldots, k$, then $\sum_{i=1}^k Y_i \sim \chi^2\left(\sum_{i=1}^k n_i, \sum_{i=1}^k \gamma_i\right)$.

c) If $Y \sim \chi^2(n, \gamma)$ then

$E Y = n + 2\gamma$ and $V(Y) = 2n + 8\gamma$.

proof b) i) $M_{\sum_{i=1}^k Y_i}(t) = \prod_{i=1}^k m_{Y_i}(t) =$

$$\prod_{i=1}^K (1-2t)^{-n_i/2} \exp\left(-\gamma_i \left[1-(1-2t)^{-1}\right]\right)$$

$$= (1-2t)^{-\frac{\sum_{i=1}^k n_i}{2}} \exp\left(-\sum_{i=1}^k \gamma_i \left[1-(1-2t)^{-1}\right]\right)$$

the $\chi^2\left(\sum_{i=1}^k n_i, \sum_{i=1}^k \gamma_i\right)$ mgf.

ii) Let $Y_i = \underline{Z}_i^T \underline{Z}_i$ where $\underline{Z}_i$ are ind. $N_{n_i}(\underline{\mu}_i, I_{n_i})$ RVs.

Then $\underset{\sim}{Z} = \begin{pmatrix} \underset{\sim}{Z}_1 \\ \underset{\sim}{Z}_2 \\ \vdots \\ \underset{\sim}{Z}_k \end{pmatrix} \sim N_{\sum_{i=1}^k n_i}\left[\underbrace{\begin{pmatrix} \underset{\sim}{\mu}_1 \\ \vdots \\ \underset{\sim}{\mu}_k \end{pmatrix}}_{\underset{\sim}{\mu}_Z}, I_{\sum_{i=1}^k n_i}\right]$

So $\underset{\sim}{Z}^T \underset{\sim}{Z} = \sum_{i=1}^k \underset{\sim}{Z}_i^T \underset{\sim}{Z}_i = \sum_{i=1}^k Y_i \sim \chi^2\left(\sum_{i=1}^k n_i, \gamma_Z\right)$

where $\gamma_Z = \frac{\underset{\sim}{\mu}_Z^T \underset{\sim}{\mu}_Z}{2} = \sum_{i=1}^k \frac{\underset{\sim}{\mu}_i^T \underset{\sim}{\mu}_i}{2} = \sum_{i=1}^k \gamma_i$.

c) i) Let $X \sim \chi^2(1, \gamma)$ $\perp\!\!\!\perp$ $W \sim \chi^2_{n-1} \sim \chi^2(n-1, 0)$

Then by b) $Y = X + W \sim \chi^2(n, \gamma)$.

Let $Z \sim N(0, 1)$, and $\delta = 2\gamma$. Then $\sqrt{\delta} + Z \sim N(\sqrt{\delta}, 1)$

and $X = (\sqrt{\delta} + Z)^2$.

Hence $E(X) = E(\sqrt{\delta} + Z)^2 =$

$\delta + 2\sqrt{\delta}\, EZ + EZ^2 = \delta + 1 = EX$

and $EX^2 = E(\sqrt{\delta} + Z)^4 =$

$a = \sqrt{\delta}, b = Z$

$(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}$

by bin th

$$E\left[\delta^2 + 4\delta^{3/2} z + 6\delta z^2 + 4\sqrt{\delta} z^3 + z^4\right]$$

but $E z = E z^3 = 0$, $E z^2 = 1$, $E z^4 = 3$

HW 2 #4

So $E X^2 = \delta^2 + 6\delta + 3$ and

$$V(X) = EX^2 - (EX)^2 = \delta^2 + 6\delta + 3 - (\delta+1)^2$$

$$= \delta^2 + 6\delta + 3 - \delta^2 - 2\delta - 1 = \underline{4\delta + 2 = V}$$

So $E(Y) = EX + EW = \delta + 1 + n - 1 = n + \delta$

$= n + 2\gamma$ and

$$V(Y) = V(X+W) = V(X) + V(W) =$$

$$4\delta + 2 + 2(n-1) = 8\gamma + 2n$$

ii) $Y \sim \chi^2(n, \gamma) \Rightarrow Y = \underset{\sim}{z}^T \underset{\sim}{z} = \sum_{i=1}^n z_i^2$

where $\underset{\sim}{z} \sim N_n(\underset{\sim}{\mu}, I_n)$

So $EY = \sum_{i=1}^n E(z_i^2) = \sum_{i=1}^n (\mu_i^2 + 1) = n + \mu^T\mu$
$= n + 2\gamma$

$z_i \sim N(\mu_i, 1)$, $E z_i^2 = 1 + \mu_i^2$

and $V(Y) = V\left(\sum_{i=1}^n z_i^2\right) = \sum_{i=1}^n V(z_i^2)$

$= \sum_{i=1}^n \left[E z_i^4 - (E z_i^2)^2\right]$

$Y \sim N(\mu, \sigma^2)$: $EY^k = (k-1)\sigma^2 EY^{k-2} + \mu EY^{k-1}$, $z_i \sim N(\mu_i, 1)$

HW2 #4

$E z_i = \mu_i$, $E z_i^2 = \mu_i^2 + 1$, $\sigma^2 = 1$

$E z_i^3 = 2\sigma^2 E z_i + \mu_i E z_i^2$

$= 2\mu_i + \mu_i(\mu_i^2 + 1) = \mu_i^3 + 3\mu_i$

$E z_i^4 = 3\sigma^2 E z_i^2 + \mu_i E(z_i^3)$

$= 3(\mu_i^2 + 1) + \mu_i\left[\mu_i^3 + 3\mu_i\right]$

$= 3\mu_i^2 + 3 + \mu_i^4 + 3\mu_i^2 = \mu_i^4 + 6\mu_i^2 + 3$

$V(Y) = \sum_{i=1}^n \left[\mu_i^4 + 6\mu_i^2 + 3 - \mu_i^4 - 2\mu_i^2 - 1\right]$

$= \sum_{i=1}^n \left[4\mu_i^2 + 2\right] = 2n + 4\underbrace{\mu^T\mu}_{2\delta}$

$= 2n + 8\delta$

23) Th. If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \Sigma)$ with $\Sigma > 0$, then

$\underset{\sim}{Y}^T A \underset{\sim}{Y} \sim \chi^2\left(\text{rank}(A), \tfrac{1}{2}\mu^T A \mu\right)$ iff

$A\Sigma$ is idempotent. If $\Sigma = I_n$ the result holds iff A is idempotent.

Note: $A = A^T$ since $Y^T A Y$ is a quadratic form.

rank A = rank $A\Sigma$ ...

24] i) If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{0}, I_n)$

$\underline{Y}^T A \underline{Y} \sim \chi^2_r$ iff A is idempotent of rank r, $= tr(A)$.

(ii) If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{0}, \Sigma)$ with $\Sigma > 0$, $\underline{Y}^T A \underline{Y} \sim \chi^2_r$

iff $A\Sigma$ is idempotent of rank r.

(iii) If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \sigma^2 I_n)$, then

$$\frac{\underline{Y}^T\underline{Y}}{\sigma^2} \sim \chi^2\left[n, \frac{\frac{1}{2}\underline{\mu}^T\underline{\mu}}{\sigma^2}\right].$$

$(\frac{1}{\sigma}\underset{\sim}{Y} \sim N_N(\frac{\underline{\mu}}{\sigma}, I_n))$

(iv) If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, I_n)$, $\underline{Y}^T A \underline{Y} \sim \chi^2\left(r, \frac{\underline{\mu}^T A \underline{\mu}}{2}\right)$

iff A is idempotent of rank r

25]* Th: If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \Sigma)$, then

$\underline{Y}^T A \underline{Y} \perp\!\!\!\perp B\underline{Y}$ iff $B\Sigma A = 0$.
iff $A\Sigma B^T = 0$

Note: do not need $\underline{Y}^T A \underline{Y}$ to have a non central χ^2 distribution in 25]

or 26]

26) If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \Sigma)$ and $\Sigma > 0$,

$\underline{Y}^T A \underline{Y} \perp\!\!\!\perp \underline{Y}^T B \underline{Y}$ iff

$$\underbrace{A \Sigma B = 0 \quad \text{iff} \quad B \Sigma A = 0}_{\text{equivalent since } A, B, \Sigma \text{ are symmetric}}$$

{Do this after 27} below.

28) If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \sigma^2 I)$ then

$\frac{1}{\sigma^2} \underline{Y}^T A \underline{Y} \sim \chi^2(\text{rank}(A), \frac{1}{2\sigma^2} \underline{\mu}^T A \underline{\mu})$

iff A is idempotent,

$(\frac{1}{\sigma} \underset{\sim}{Y} \sim N_n(\frac{\underline{\mu}}{\sigma}, I))$

27) * $\underline{\text{Craig's th for } \Sigma \geq 0.}$

If $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \Sigma)$ where $\Sigma \geq 0$, then

a) $\underline{Y}^T A \underline{Y} \perp\!\!\!\perp \underline{Y}^T B \underline{Y}$ iff

(*) $\Sigma A \Sigma B \Sigma = \overset{\text{matrix}}{0}$, $\Sigma A \Sigma B \underset{\sim}{\mu} = \underset{\sim}{0}$, $\Sigma B \Sigma A \underset{\sim}{\mu} = \underset{\sim}{0}$

$\underline{\mu}^T A \Sigma B \underline{\mu} = \underset{\text{scalar}}{0}$

b) If $\Sigma \geq 0$ $\quad Y^T A Y \perp\!\!\!\perp Y^T B Y$ if $A \Sigma B = 0$ $\Leftarrow$ sufficient for $\Sigma \geq 0$

necessary and sufficient if $\Sigma > 0$

Chr

Note that if $A \Sigma B = 0$ then (*) holds.

LM) 13 p336 $Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_{p-1} x_{i,p-1} + \varepsilon_i$

for $i = 1, \ldots, n$ or (17.5)

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} x_{10} & x_{11} & x_{12} \cdots & x_{1,p-1} \\ x_{20} & x_{21} & x_{22} \cdots & x_{2,p-1} \\ \vdots & & & \\ x_{n0} & x_{n1} & x_{n2} \cdots & x_{n,p-1} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

or $\underset{\sim}{Y} = \mathbb{X} \underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$ where $x_{i0} \equiv 1$.

design matrix $\underset{n \times p}{\mathbb{X}}$ has full rank p.

$\mathbb{X}$ and $\underset{\sim}{\beta}$ are treated as a constant matrix and vector, $\underset{\sim}{Y}$ and $\underset{\sim}{\varepsilon}$ are random vectors. Condition of $\mathbb{X}$ if $\mathbb{X}$ is a random matrix.

23 p336 A linear regression model, is linear in the parameters $\underset{\sim}{\beta}$ so

$Y_i = \underset{\sim}{w}_i^T \underset{\sim}{\beta} + \varepsilon_i$ is a linear model,