$F_R > F_{q,n-p,1-\delta}$ is a large sample right tail δ test for the OLS model for a large class of zero mean error distributions.

9) If $X \sim t_{n-p}$ then $X^2 \sim F_{1,n-p}$.

The wald ~~t~~ test $H_0\ \beta_j = 0$ $\underbrace{H_1\ \beta_j \neq 0}$ $\overset{\text{2 tail t test}}{}$

is equivalent to the right tail F test

since rejecting H_0 if $|X| > t_{n-p,1-\delta}$

is equivalent to rejecting H_0 if $X^2 > F_{1,n-p,1-\delta}$.

10) For a right tailed test with test statistic T_n, the <u>pvalue</u> =

P_{H_0}[of observing a test statistic $\geq$ the test statistic actually observed].

Under the OLS model where

$F_R \overset{H_0}{\sim} F_{q,n-p}$ (so the e_i are iid $N(0,\sigma^2)$),

pvalue $= P(W > F_R)$ where $W \sim F_{q,n-p}$.

In general can only estimate pvalue.

Let pval be the estimated pvalue.

Then if H_0 is true

pval $= P(W > F_R)$ where $W \sim F_{q,n-p}$

and pval $\xrightarrow{P}$ pvalue as $n \to \infty$ for the large sample partial F test.

The pvalues in output are usually pvals (estimated pvalues).

11) Want $n \geq 10p$ before doing inference (tests, confidence intervals, prediction). Need much larger n if the error distribution is not close to $N(0,\sigma^2)$.

p100 Know for Q5 LM 35

12) ✓ Proof that $F_R \sim F_{g,n-p}$ if $\underset{\sim}{\varepsilon} \sim N_n(\underline{0}, \sigma^2 \underline{I})$

Let $\underline{Y} = \underline{X}\underline{\beta} + \underset{\sim}{\varepsilon}$ $\quad E(\underset{\sim}{\varepsilon}) = \underline{0}$, $Cov(\underset{\sim}{\varepsilon}) = \sigma^2 \underline{I}$

Let $\underline{X} = (\underline{X}_1 \ \underline{X}_2)$, $\underline{\beta} = \begin{pmatrix} \underline{\beta}_1 \\ \underline{\beta}_2 \end{pmatrix}$

$H_0: \underline{\beta}_2 = \underline{0}$ so the reduced model is

$\underline{Y} = \underline{X}_1 \underline{\beta} + \underset{\sim}{\varepsilon}$. Let $\underline{P}$ be the projection matrix on $C(\underline{X})$ and $\underline{P}_1$ the projection matrix on $C(\underline{X}_1) \subset C(\underline{X})$.

a) $\underline{P}_1 \underline{P}_1 = \underline{P}_1 = \underline{P}_1 \underline{P}$

b) $F_R = \frac{n-p}{g} \frac{\underset{\sim}{Y}'(\underline{P} - \underline{P}_1)\underset{\sim}{Y}}{\underline{Y}'(\underline{I} - \underline{P})\underline{Y}} \sim F_{g,n-p}$

proof) a) $\underline{P}\,\underline{P}_1 = \underline{P}_1$ since $C(\underline{P}_1) = C(\underline{X}_1) \subseteq C(\underline{X}) = C(\underline{P})$

so the columns of $\underline{P}_1 \in C(\underline{X})$.

Transposing gives $\underline{P}_1 = \underline{P}_1 \underline{P}$.

b) $F_R = \dfrac{\dfrac{RSS(R) - RSS(F)}{q}}{RSS(F)/(n-p)}$.

$RSS(R) = (\underline{Y} - \hat{\underline{Y}}_1)'(\underline{Y} - \hat{\underline{Y}}_1) = [(\underline{I} - \underline{P}_1)\underline{Y}]'(\underline{I} - \underline{P}_1)\underline{Y}$

$= \underline{Y}'(\underline{I} - \underline{P}_1)\underline{Y}$. Similarly

$RSS(F) = \underline{Y}'(\underline{I} - \underline{P})\underline{Y}$. So

$RSS(R) - RSS(F) = \underline{Y}'[\underline{I} - \underline{P}_1 - (\underline{I} - \underline{P})]\underline{Y}$

$= \underline{Y}'(\underline{P} - \underline{P}_1)\underline{Y}$ and

$F_R = \dfrac{n-p}{q} \dfrac{\underline{Y}'(\underline{P} - \underline{P}_1)\underline{Y}}{\underline{Y}'(\underline{I} - \underline{P})\underline{Y}} \dfrac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2}}$

Now $(\underline{P} - \underline{P}_1)(\underline{P} - \underline{P}_1) = \underline{P} - \underline{P}_1 - \underline{P}_1 + \underline{P}_1 = \underline{P} - \underline{P}_1$

so $\underline{P} - \underline{P}_1$ is symmetric and idempotent

as is $\underline{I} - \underline{P}$. $(\underline{P} - \underline{P}_1)(\underline{I} - \underline{P}) = \underline{P} - \underline{P} - \underline{P}_1 + \underline{P}_1 = 0$

so $\dfrac{\underline{Y}'(\underline{P} - \underline{P}_1)\underline{Y}}{\sigma^2} \overset{p101}{\sim} \chi^2_{d_1} \perp\!\!\!\perp \dfrac{\underline{Y}'(\underline{I} - \underline{P})\underline{Y}}{\sigma^2} \overset{p101}{\sim} \chi^2_{d_2}$.

$\text{rank}(\underline{I} - \underline{P}) = tr(\underline{I} - \underline{P}) = n - p = d_2$

(also see Q8 #2)

$\text{rank}(\underset{\sim}{P} - \underset{\sim}{P}_1) = tr(\underset{\sim}{P} - \underset{\sim}{P}_1) = p - k - 1 = q = d_1$

So $F_R = \dfrac{X_1/q}{X_2/(n-p)} \sim F_{q,n-p}$ where

$X_1 \sim \chi^2_q \perp\!\!\!\perp X_2 \sim \chi^2_{n-p}$.

13) Let $\underset{\sim}{x}_i = \begin{pmatrix} 1 \\ \underset{\sim}{u}_i \end{pmatrix} = \begin{pmatrix} 1 \\ x_{i,1} \\ \vdots \\ x_{i,p-1} \end{pmatrix}$

So $\underset{\sim}{X} = (\underset{\sim}{1} \;\; \underset{\sim}{X}_1) = \begin{bmatrix} 1 & x_{1,1} & x_{12} & \cdots & x_{1,p-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n1} & x_{n2} & & x_{n,p-1} \end{bmatrix}$

Let $\frac{1}{n}\sum_{j=1}^{n} x_{jk} = \underbrace{\overline{x}_{0k}}_{} = \overline{u}_{0k}$ $\quad k = 1, \ldots, p-1$

sum over j

Let $\overline{\underset{\sim}{u}} = (\overline{u}_{01}, \ldots, \overline{u}_{0p-1})$

Let the <u>sample covariance matrix</u>

$$\underset{(p-1)\times(p-1)}{\hat{\Sigma}_{\underset{\sim}{u}}} = \frac{1}{n-d_1}\sum_{i=1}^{n}(\underset{\sim}{u}_i - \overline{\underset{\sim}{u}})(\underset{\sim}{u}_i - \overline{\underset{\sim}{u}})' = \widehat{Cov}(\underset{\sim}{u})$$

Let the <u>sample covariance matrix</u> $\widehat{Cov}(\underset{\sim}{u}, Y)$

$$= \underset{(p-1)\times 1}{\hat{\Sigma}_{\underset{\sim}{u}Y}} = \frac{1}{n-d_2}\sum_{i=1}^{n}(\underset{\sim}{u}_i - \overline{\underset{\sim}{u}})\underbrace{(Y_i - \overline{Y})}_{\text{scalar}}$$

usually $d_i = 0$ or 1.

If $d_2 = 0$, $\hat{\Sigma}_{\underset{\sim}{U}Y} = \frac{1}{n} \sum_{i=1}^{n} (\underset{\sim}{U}_i - \bar{\underset{\sim}{U}})(Y_i - \bar{Y})$

$$= \frac{1}{n} \sum_{i=1}^{n} \underset{\sim}{U}_i Y_i - \bar{\underset{\sim}{U}} \bar{Y} \qquad \left(\sum_{i=1}^{n} (Y_i - \bar{Y}) = 0 \right)$$

If $\underset{\sim}{W}_i = \begin{pmatrix} Y_i \\ \underset{\sim}{U}_i \end{pmatrix}$ are iid such that $\Sigma_{\underset{\sim}{W}} = \begin{pmatrix} \sigma_Y^2 & \Sigma_{Y\underset{\sim}{U}} \\ \Sigma_{\underset{\sim}{U}Y} & \Sigma_{\underset{\sim}{U}} \end{pmatrix}$ exists,

then $\hat{\Sigma}_{U} \xrightarrow{P} \text{cov}(\underset{\sim}{U}) = \Sigma_{\underset{\sim}{U}}$ and $\hat{\Sigma}_{\underset{\sim}{U}Y} \xrightarrow{P} \text{cov}(\underset{\sim}{U}, Y) = \Sigma_{\underset{\sim}{U}Y}$.

Ferguson p 57 2nd moments CLT needs 4th moments

14) ✓ p106 Let $X = (\underset{\sim}{1} \ X_1)$. Then $(X^T X) = \begin{pmatrix} n & n\bar{\underset{\sim}{U}}^T \\ n\bar{\underset{\sim}{U}} & X_1^T X_1 \end{pmatrix}$

$$(X^T X)^{-1} = \begin{pmatrix} \frac{1}{n} + \bar{\underset{\sim}{U}}^T D^{-1} \bar{\underset{\sim}{U}} & -\bar{\underset{\sim}{U}}^T D^{-1} \\ -D^{-1} \bar{\underset{\sim}{U}} & D^{-1} \end{pmatrix}$$

where $D^{-1} = \left[(n-1) \hat{\Sigma}_{\underset{\sim}{U}} \right]^{-1} = \frac{\hat{\Sigma}_{\underset{\sim}{U}}}{n-1}$ where $d_1 = 1$.

(Seber uses $D = V$.)

15) ✓ $\hat{\underset{\sim}{\beta}} = (X'X)^{-1} X'Y = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\underset{\sim}{\beta}}_S \end{pmatrix}$

where $\hat{\underset{\sim}{\beta}}_S = \begin{pmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_{p-1} \end{pmatrix}$.

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16} Th If $\underset{\sim}{w}_i = \begin{pmatrix} Y_i \\ \underset{\sim}{u}_i \end{pmatrix}$ are iid, $\Sigma_{\underset{\sim}{w}} = \begin{pmatrix} \sigma_Y^2 & \Sigma_{Y\underset{\sim}{u}} \\ \Sigma_{\underset{\sim}{u}Y} & \Sigma_{\underset{\sim}{u}} \end{pmatrix}$ exists and $\Sigma_{\underset{\sim}{u}}^{-1}$ exists, then

$$\hat{\beta}_0 = \bar{Y} - \hat{\underset{\sim}{\beta}}_S^T \bar{\underset{\sim}{u}} \xrightarrow{P} E(Y) - \underset{\sim}{\beta}_S^T E[\underset{\sim}{u}] = \beta_0$$

$$\hat{\underset{\sim}{\beta}}_S = \frac{n}{n-1} \underbrace{\hat{\Sigma}_{\underset{\sim}{u}}^{-1}}_{d_1=1} \underbrace{\hat{\Sigma}_{\underset{\sim}{u}Y}}_{d_2=0} \xrightarrow{P} \underset{\sim}{\beta}_S = \Sigma_{\underset{\sim}{u}}^{-1} \Sigma_{\underset{\sim}{u}Y}.$$

proof}

Note that $\underset{\sim}{Y}^T X_1 = (Y_1 \cdots Y_n) \begin{pmatrix} \underline{u}_1^T \\ \vdots \\ \underline{u}_n^T \end{pmatrix} = \sum_{i=1}^n Y_i \underline{u}_i^T$

and $X_1^T \underline{Y} = [\underline{u}_1 \cdots \underline{u}_n] \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \sum_{i=1}^n \underline{u}_i Y_i$

$$\text{so } \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_S \end{pmatrix} = \begin{bmatrix} \frac{1}{n} + \bar{\underline{u}}^T D^{-1} \bar{\underline{u}} & -\bar{\underline{u}}^T D^{-1} \\ -D^{-1} \bar{\underline{u}} & D^{-1} \end{bmatrix} \begin{pmatrix} \underset{\sim}{1}^T \\ X_1^T \end{pmatrix} \underset{\sim}{Y}$$

$$= \begin{bmatrix} \frac{1}{n} + \bar{\underline{u}}^T D^{-1} \bar{\underset{\sim}{u}} & -\bar{\underset{\sim}{u}}^T D^{-1} \\ -D^{-1} \bar{\underset{\sim}{u}} & D^{-1} \end{bmatrix} \begin{pmatrix} n\bar{Y} \\ X_1^T \underline{Y} \end{pmatrix}$$

So $\hat{\beta}_S = -nD^{-1}\bar{\underset{\sim}{u}}\,\bar{Y} + D^{-1}X_1^T \underline{Y}$

$= D^{-1}\left(X_1^T\underline{Y} - n\bar{\underline{u}}\,\bar{Y}\right) =$

$D^{-1}\left[\sum_{i=1}^{n} \underset{\sim}{u}_i Y_i - n\bar{\underline{u}}\,\bar{Y}\right]$

$= \frac{\hat{\Sigma}_{\underline{u}}^{-1}}{n-1}\, n\, \hat{\Sigma}_{\underline{u}Y} = \frac{n}{n-1}\hat{\Sigma}_{\underset{\sim}{u}}^{-1}\hat{\Sigma}_{\underline{u}Y}$

$$\text{and } \hat{\beta}_0 = \bar{Y} + \underbrace{n\,\bar{\underline{u}}^T D^{-1}\bar{\underline{u}}\,\bar{Y}} - \overbrace{\bar{\underline{u}}^T D^{-1}X_1^T\underset{\sim}{Y}}^{\text{scalar } a^T = a}$$

$= \bar{Y} + \left[n\bar{Y}\,\bar{\underline{u}}^T D^{-1} - \underline{Y}^T X_1 D^{-1}\right]\bar{\underset{\sim}{u}}$

$= \bar{Y} - \hat{\underset{\sim}{\beta}}_S^T\,\bar{\underset{\sim}{u}}.$

Note: this result was also shown

where $\begin{pmatrix} Y \\ X_1 \\ \vdots \\ X_p \end{pmatrix} \sim N_{p+1}\left[\begin{pmatrix} E(Y) \\ E(\underline{X}) \end{pmatrix}, \begin{pmatrix} \sigma_Y^2 & \Sigma_{Y\underline{X}} \\ \Sigma_{\underline{X}Y} & \Sigma_{\underline{X}\underline{X}} \end{pmatrix}\right]$

after [illegible] in ch 2

Where $E(Y \mid \underline{x} = \underline{x}) = \alpha + \underline{\beta}^T \underline{x}$

$\beta_0 = \alpha = E(Y) - \underline{\beta}^T E(\underline{x})$ and

$\underline{\beta}_S = \underline{\beta}_2 = \Sigma_{xx}^{-1} \Sigma_{xy}$.

Searle p60-1 Generalized Cochran's

18) **Theorem:** Let $\underset{\sim}{Y} \sim N_n(\underline{\mu}, \Sigma)$, let

$\underset{n \times n}{A_i} = A_i'$ have rank r_i, $i = 1, \ldots, k$

and let $A = \sum_{i=1}^{k} A_i = A'$ have rank r.

Then $\underline{Y}' A_i \underline{Y} \sim \chi^2(r_i, \frac{1}{2} \underline{\mu}' A_i \underline{\mu})$

and the $\underline{Y}' A_i \underline{Y}$ are independent

and $\underline{Y}' A \underline{Y} \sim \chi^2(r, \frac{1}{2} \underline{\mu}' A \underline{\mu})$

iff I) any 2 of a) $A_i \Sigma$ are idempotent $\forall i$

b) $A_i \Sigma A_j = 0$ $\forall i < j$

c) $A \Sigma$ is idempotent

are true or

II) c) is true and d) $r = \sum_{i=1}^{k} r_i$

or III) c) is true and

e) $A_1 \Sigma, \ldots, A_{k-1} \Sigma$ are idempotent

and $A_k \Sigma \geq 0$ is singular.

19] Often $A = \sum_{i=1}^{k} A_i = I_n$.

20] Christensen P33 Distribution of F_R under normality when H_0 may not hold

Assume $\underset{\sim}{Y} = X \underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$, $\underset{\sim}{\varepsilon} \sim N_n(\underset{\sim}{0}, \sigma^2 I)$

Let $X = [X_1 \; X_2]$ be full rank and

let the reduced model $\underline{Y} = X_1 \underline{\beta_1} + \underset{\sim}{\varepsilon}$. Then

$$F_R = \frac{\underline{Y}'(P - P_1)\underline{Y}/q}{\underline{Y}'(I - P)\underline{Y}/(n-p)} \sim F\left(q, n-p, \frac{\underset{\sim}{\beta}' X'(P - P_1) X \underset{\sim}{\beta}}{2\sigma^2}\right)$$

where $F(d_1, d_2, \gamma)$ is a noncentral F distribution

with d_1 and d_2 numerator and denominator degrees

of freedom and noncentrality parameter γ.