practice for Exam I

Math 588 Quiz 2 Fall 2012 Name___________

1) Suppose that

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_4\left(\begin{pmatrix} 49 \\ 25 \\ 9 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 3 & 0 \\ -1 & 5 & -3 & 0 \\ 3 & -3 & 5 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \right).$$

a) Find the distribution of $(X_1, X_3)^T$.

$$N_2\left[\begin{pmatrix} 49 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\right]$$

b) Which pairs of random variables X_i and X_j are independent?

$X_1 \perp\!\!\!\perp X_4$, $X_2 \perp\!\!\!\perp X_4$, $X_3 \perp\!\!\!\perp X_4$

c) Find the correlation $\rho(X_1, X_3)$.

$$\frac{\operatorname{cov}(X_1, X_3)}{\sqrt{V(X_1)\,V(X_3)}} = \frac{3}{\sqrt{2}\sqrt{5}} = \boxed{\frac{3}{\sqrt{10}} = 0.9487}$$

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2) Recall that if $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$, then the conditional distribution of $\boldsymbol{X}_1$ given that $\boldsymbol{X}_2 = \boldsymbol{x}_2$ is multivariate normal with mean $\boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)$ and covariance matrix $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Let Y and X follow a bivariate normal distribution

$$\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2\left(\begin{pmatrix} 49 \\ 17 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix} \right).$$ $\mu_y + \Sigma_{YX}\Sigma_{XX}^{-1}(X - \mu_X)$

a) Find $E(Y|X)$. $49 + (-1)\frac{1}{4}(x_2 - 17)$

$= 49 - \frac{1}{4}(x_2 - 17) = 49 + \frac{17}{4} - \frac{1}{4}x_2 = \boxed{53.25 - 0.25x_2}$

b) Find $\text{Var}(Y|X)$.

$3 - (-1)\frac{1}{4}(-1) = 3 - \frac{1}{4} = \boxed{2.75} = \frac{11}{4}$

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3) Suppose $\boldsymbol{Y} \sim N_n(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{I})$. Find the distribution of $Z =$ $(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}$ if $\boldsymbol{X}$ is an $n \times p$ full rank constant matrix and $\boldsymbol{\beta}$ is a $p \times 1$ constant vector. Simplify

$$E\underline{Z} = (X^TX)^{-1}X^TX\underline{\beta} = \underset{\sim}{\beta}$$

$$\operatorname{COV}(\underline{Z}) = (X^TX)^{-1}X^T\sigma^2 I \; X (X^TX)^{-1}$$

$$= \sigma^2 (X^TX)^{-1} X^TX (X^TX)^{-1}$$

$$= \sigma^2 (X^TX)^{-1}$$

$$\text{So } \underset{\sim}{Z} \sim N_p\left[\underline{\beta}, \sigma^2(X^TX)^{-1}\right]$$

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