MATH 530– Dr. Z. Ye

# Final Exam

Fall 1993

1. (40 pts) True-False. If the assertion is true, quote a relevant theorem or reason, or give a proof; if false, give a counterexample or other justification.

   a) If $f$ is a bounded function in $[0,\ 1]$, then $f$ is measurable in $[0,\ 1]$.

   b) If $O$ is an open set in $\mathbf{R}$, then $m(O)$ is positive.

   c) Any continuous function $f$ in an interval $I$ is a measurable function in $I$.

   d) Let $f$ be bounded in $[0,\ 1]$. If $f$ is measurable over $[0,\ 1]$, then $f$ is continuous at least at one point $x_0$ with $x_0 \in [0,\ 1]$.

   e) Let $f$ be a measurable in a set $E$. If $|f|$ is integrable over $E$, then $f$ is integrable over $E$.

   f) Let $E$ be a bounded set in $\mathbf{R}$, then every characteristic function of the set $E$ is integrable over $\mathbf{R}$ in the sense of Lebesgue.

   g) Every absolutely continuous function on $[a,\ b]$ is differentiable almost everywhere on $[a,\ b]$.

   h) Every absolute continuous function in $[0,\ 1]$ is uniformly continuous in $[0,\ 1]$.

2. (20 pts) Assume a set $A$ is measurable and $m(A) \leq \infty$. Set $\phi(x) = m(A \cap (-\infty,\ x])$. Prove.

   (i) $\phi$ is continuous in $\mathbf{R}$.

   (ii) If $A$ is a bounded set with $m(A) > 0$, then, for any $\eta$ with $0 < \eta < m(A)$, there exists a set $B \subset A$ such that $m(B) = \eta$.

3. (25 pts) Let $f$ be integrable over $E$. Prove.

$$(i)\quad \lim_{t\to\infty} m(\{x \in E : \ |f| \geq t\}) = 0.$$

$$(ii)\qquad \lim_{t\to\infty} \int_{\{x\in E:\ |f|\geq t\}} |f| = 0.$$

4. (20 pts) Let $\mathbf{Q}$ be a set of rational numbers in $\mathbf{R}$. Let

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q} \cap [0.\ 1], \\ \ln(1+x), & \text{if } x \in [0,\ 1] \setminus \mathbf{Q}. \end{cases}$$

   Find $(R)\int_0^1 f(x)\,dx$ and $\int_{[0,\ 1]} f(x)\,dx$ if they exist.

5. (30 pts) Let $p \in (1,\ \infty)$. Let $f_n$ be a sequence of functions such that $f_n \to f$ a.e. with $f \in L_p$ and $f_n \in L_p$. Then $||f_n - f||_p \to 0$ if and only if $||f_n||_p \to ||f||_p$, as $n \to \infty$.

Fin 93

6. (25 pts) Let

$$f(x) = \int_0^\infty \sin(x^2 + t^2)e^{-t}\, dt.$$

Prove $f$ is continuous in $\mathbf{R}$.

7. (20 pts) Let $f$ be continuous and differetiable in $[0,\ 1]$. If there is a positive constant $M$ such that $|f'(x)| \leq M$ for all $x \in [0,\ 1]$, then $f$ is absolutely continuous in $[0,\ 1]$.

8. (20 pts) Prove $\int_0^\pi x^{-1/4} \sin x\, dx \leq \pi^{3/4}$.

Fa 93

# Answers to Final

1. (a) False. Let $P$ be a non-measurable set in $[0,1]$. Set
$$f(x) = \begin{cases} 1, & x \in P \\ 0, & x \notin P \end{cases}$$
then $f$ is bounded but is not measurable.

(b) False. Empty set $\phi$ is open but $m(\phi) = 0$

(c) True. Since, for any $\alpha$, the set $\{x \in I, f(x) > \alpha\}$ is open set so it is measurable set. Hence $f$ is measurable

(d) False. Look at
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ -1 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

(e) True. Because
$$\left| \int_E f \right| \le \int_E |f| < \infty$$

(f) False. Let $P$ be a non-measurable set in $[0,1]$. $\chi_P$ is bounded but is not integrable since $\chi_P$ is not measurable

(g) True. $f$ is A.C. $\Rightarrow$ $f$ is BV. $\Rightarrow$ $f = g - h$, where $g \nearrow$, $h \nearrow$ $\Rightarrow$ $f$ is differentiable almost everywhere since monotone function is differentiable a.e.

(h) True. By definitions.

2. (i) For any $x_0 \in \mathbb{R}$, $\forall \varepsilon > 0$, take $\delta = \varepsilon$, then $|\varphi(x) - \varphi(x_0)| \le |x - x_0| < \varepsilon$ whenever $|x - x_0| < \delta$.

(ii) Since $\phi(x)$ is continuous and by Intermediate-Value Theorem, there exists $x_0 \in \mathbb{R}$ s.t. $\phi(x_0) = \eta$. Set $B = A \cap (-\infty, x_0]$, we have $m(B) = \phi(x_0) = \eta$.

$f$ is integrable so $\int_E |f| = M < \infty$

3 (i)
$$M = \int_E |f|\,dx \ge \int_{\{x \in E;\ |f(x)| > t\}} |f|\,dx \ge \int_{\{x \in E;\ |f| > t\}} t\,dx \ge t\,m\{x \in E, |f| > t\}, \text{ for any } t > 0$$
hence $m\{x \in E, |f| > t\} \le \frac{M}{t} \to 0$, as $t \to \infty$.

(ii) Since $|f|$ is integrable, $\forall \varepsilon$, $\exists \delta$ s.t. $\int_A |f| < \varepsilon$ for $|A| < \delta$. By (i), $\exists T_0$ s.t. $t \ge T_0$ $m\{x \in E;\ |f| > t\} < \delta$, hence $\int_{\{x \in E;\ |f| > t\}} |f| < \varepsilon$, for all $t \ge T_0$.

Fall 93

4. $f$ is not Riemann integrable since $m(\{x \in [0,1];\ f(x) \text{ is not conti. at } x\}) = 1 >$
Since $\int_{[0,1]} f\,dx = \int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1$, so $f$ is Lebesgue integrable

see Ash P 83

5. "$\Rightarrow$" Since $|f_n|^p \le 2^p(|f_n - f|^p + |f|^p) \equiv g_n(x)$ and $|f_n|^p \to |f|^p$, $g_n(x) \to 2^p|f|^p$,
so by L.C.T. (Pg 92), $\int |f|^p = \lim \int |f_n|^p \Rightarrow \|f_n\|_p \to \|f\|_p$.
"$\Leftarrow$" Since $|f_n - f|^p \le 2^p(|f_n|^p + |f|^p) \equiv g_n(x)$, and $|f_n - f|^p \to 0$, $g_n(x) \to 2^p|f|^p$
so by L.C.T, we have $\|f_n - f\|_p \to 0$

6. For any given $x_0 \in \mathbb{R}$, $\forall x_n \in \mathbb{R}$, with $x_n \to x_0$, we show $f(x_n) \to f(x_0)$.
Indeed. $f(x_n) = \int_0^\infty \sin(x_n^2 + t^2)e^{-t}\,dt$, set $f_n(t) = \sin(x_n^2 + t^2)e^{-t}$,
then $|f_n(t)| \le e^{-t}$ and $\int_0^\infty e^{-t}\,dt < \infty$, and $f_n(t) \to \sin(x_0^2 + t^2)e^{-t}$. By L.C.T on Pg 91, $\lim f(x_n) = \int_0^\infty \lim_{n\to\infty} f_n(t)\,dt = f(x_0)$.

7. $\forall \varepsilon$, $\exists \delta = \frac{\varepsilon}{M}$ s.t. $\sum_{i=1}^n |f(x_i) - f(x_i')| \le \sum_{i=1}^n M|x_i - x_i'| \le \varepsilon$, whenever $\sum_{i=1}^n |x_i' - x_i| < \delta$

8.
$$\int_0^\pi x^{-\frac14}\sin x\,dx \le \left(\int_0^\pi x^{-\frac12}dx\right)^{1/2}\left(\int_0^\pi \sin^2 x\,dx\right)^{1/2} \le \pi^{3/4}.$$

$$\int_0^\pi x^{-\frac12}\,dx = \frac{x^{\frac12}}{\frac12}\Big|_0^\pi = 2\sqrt{\pi}$$

$$\int_0^\pi \sin^2 x\,dx \le \pi$$

J93 1

# Examination B

## Analysis

June 1993

**Instructions:** All candidates should attempt 4 of the 5 problems in Part A. Those taking the 2-hour examination should work 4 of the 10 problems in Part B. Those taking the 3-hour examination should work 8 of the 10 problems in Part B. Clearly indicate which problems you wish to have scored.

PART A. Work 4 of the 5 problems.

A1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

(a) What does it mean to say that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers?

(b) Suppose that $|x_{n+1} - x_n| \le 1/2^n$ for all $n$. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges.

A2. Let $A$ be a non-empty subset of $\mathbf{R}$, and let $f : A \longrightarrow \mathbf{R}$ be a function.

(a) What does it mean to say that $f$ is uniformly continuous on $A$?

Determine whether each of the following functions is uniformly continous on the indicated domain:

(b) $f(x) = \dfrac{1}{1+x^2}$, $A = \mathbf{R}$ ← See Ross P108

(c) $f(x) = \dfrac{1}{x-1}$, $A = \{x : 0 < x < 1\}$

b=0

A3. Let $a$ and $b$ be real numbers, with $a > 0$. Prove that the equation $x^3 + ax + b = 0$ has exactly one real solution. Carefully state any theorems that you use in reaching your conclusion.

A4. Consider the transformation $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ defined by $f(x,y) = \left(x^2, \dfrac{y}{x^2+1}\right)$.

(a) Find the range of $f = \left\{f(x,y) : (x,y) \in \mathbf{R}^2\right\}$, and show that each point in the range of $f$ is the image under $f$ of either one or two points in $\mathbf{R}^2$.

(b) Determine the set of points in $\mathbf{R}^2$ where $f$ has a local inverse function. At such points, compute an explicit formula for the inverse function.

A5. Consider the infinite series of functions $\sum_{n=1}^{\infty} \frac{x}{n(x+n)}$.

(a) Show that the series converges pointwise on $[0, \infty)$ to a function $f(x)$. Show that the convergence is uniform on any bounded interval $[0, b]$, where $0 < b < \infty$.

(b) Let $S_N(x) = \sum_{n=1}^{N} \frac{x}{n(x+n)} = \sum_{n=1}^{N} \left[\frac{1}{n} - \frac{1}{x+n}\right]$.

Show that $\int_0^1 S_N(x)\,dx = \left[\sum_{n=1}^{N} \frac{1}{n}\right] - \ln(N+1)$.

(c) Using (a) and (b), show that $\lim_{N \to \infty} \left(\left[\sum_{n=1}^{N} \frac{1}{n}\right] - \ln(N+1)\right)$ exists and is finite.

5 93 3

PART B. Work 4 of the 10 problems for the 2-hour examination. Work 8 of the 10 problems for the 3-hour examination. Clearly indicate which problems you wish to have scored.

B1. Let $E$ be a Lebesgue measurable subset of $[0,1]$.

(a) Show that if $m(E)=1$, then $E$ must be dense in $[0,1]$.

(b) Show that if $m(E)=0$, then $E$ must have empty interior.

(c) Is the converse to either (a) or (b) true? Give reasons for your answer. Q

B2. Let $f:\mathbf{R}\longrightarrow\mathbf{R}$ be a function which is differentiable at every point.

(a) Show that $f':\mathbf{R}\longrightarrow\mathbf{R}$ is a Lebesgue measurable function.

(b) Will $f'$ always be a Lebesgue integrable function? Give reasons for your answer.

B3. Let $B$ be a Borel set in $\mathbf{R}$. Show that there is a sequence $\{F_n\}_{n=1}^{\infty}$ of closed subsets of $\mathbf{R}$ such that $B$ is a member of the smallest $\sigma$-algebra of subsets of $\mathbf{R}$ which contains all the $F_n$.

B4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued Lebesgue integrable functions defined on $\mathbf{R}$ such that for each $n$, $0\le f_{n+1}(x)\le f_n(x)$ a.e. Prove that $\{f_n\}_{n=1}^{\infty}$ converges pointwise a.e. to $f(x)=0$ if and only if $\lim_n \int_{\mathbf{R}} f_n = 0$.

$a \Rightarrow b \quad \Leftrightarrow$ not $b \Rightarrow$ not

B5. Define $f:[0,1]\longrightarrow\mathbf{R}$ by $f(x)=\begin{cases} x^2\cos\left(\frac{1}{x^2}\right), & x\ne 0\\ 0, & x=0\end{cases}$

(a) Show that $f$ is differentiable at every point of $[0,1]$, but is not a function of bounded variation on $[0,1]$.

(b) Is it true that $f(x)=\int_0^x f'(t)\,dt$ for each $x$, $0\le x\le 1$? Give reasons for your answer.

Jun 93 $\boxed{1}$

$\boxed{A1}$ a) $\{x_n\}$ is Cauchy if $\forall \varepsilon > 0$
$\exists N \ni$ for all $m,n > N$ $|x_n - x_m| < \varepsilon$.

b) Let $\varepsilon > 0$. Let $n > m$.

Now $|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m|$

$$\le \underbrace{|x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m|}_{n-m \text{ terms}}$$

$n-1$ $n-(n-m)$

$$\le \underbrace{\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m}}_{} \le \frac{n-m}{2^m}$$

tail of a convergent series as $n > m \to \infty$.

$$\sum_{j=m}^{n-1} \left(\frac{1}{2}\right)^j = \sum_{j=0}^{n-1} \left(\frac{1}{2}\right)^j - \sum_{j=0}^{m-1} \left(\frac{1}{2}\right)^j$$

$$= \frac{\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} - \frac{\left(\frac{1}{2}\right)^m}{1-\frac{1}{2}} = 2\left(\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^m\right)$$

$$(1-x)\sum_{j=0}^{m} x^j = (1-x)(1 + x + x^2 + \cdots + x^m)$$

$$= 1 + x + \cdots + x^m - x - \cdots - x^m - x^{m+1} = 1 - x^{m+1}$$

$$|x_n - x_m| \le 2\left(\frac{1}{2}\right)^m \left(\left(\frac{1}{2}\right)^{n-m} - 1\right) \le \left(\frac{1}{2}\right)^{n-1}$$

$< \varepsilon$ for $n$ large enough.

$\therefore \{x_n\}$ is Cauchy, hence convergent.

(a) $f$ is U.C. on $A$ if $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$

$\ni$ if $x_1, x_2 \in A$ with $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| \le \varepsilon$.  Jun 93

b) $$f'(x) = \frac{(1+x^2)0 - 1\,2x}{(1+x^2)^2}$$

$$= \frac{-2x}{(1+x^2)^2}$$

$$f''(x) = \frac{(1+x^2)^2(-2) - (-2x)2(1+x^2)2x}{(1+x^2)^2} \stackrel{\text{set}}{=} 0$$

If $|x| \le 1$ then $|f'(x)| \le 2$

If $|x| > 1$ then since $|-2x| \le 2\,(1+x^2)^2$

$|f'(x)| \le 2$.

$\therefore$ $|f'(x)| \le 2$

Let $a < b$ and $b - a < \frac{\varepsilon}{2}$

By the mean value theorem, for some $x_0$ in $(a, b)$

$$|f(b) - f(a)| = |f'(x_0)|\,|b-a| \le 2\,\frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ U.C.

c) Not U.C., can't be extended to a continuous function on $[0, 1]$.

or use p 87 Gaughan

Ross p 107

or th 3.9 p 96

June 1993

A3 $x = \frac{-b}{x^2+a} < 0$ $\boxed{2\frac{1}{2}}$

$\Rightarrow x < 0$ $\Rightarrow$ there is at most one solution

If $b > 0$ $x(x^2+a) = -b \Rightarrow x > 0$

$\therefore$ there is at most one solution if $b > 0$.

If $b = 0$ get $x(x^2+a) = 0$

$\Rightarrow x = 0$ $x = \pm i\sqrt{a}$

Now if $b > 0$, $a > 0$

$x = \frac{-b}{x^2+a}$ has a real solution

because

4 each

A4 a) Fix $x = 1$ then $\frac{y}{x}$ ranges over $\mathbb{R}$

$\therefore$ range of $f = \{(x, y) \ni x \geq 0, y \in \mathbb{R}\}$
$= A$

Let $(a, b) \in A$

then $a = x^2$ $b = \frac{y}{x^2+1}$

$a = \pm x$ $b = \frac{y}{(\pm x)^2+1}$

$\therefore f\left(+x, \frac{y}{(+x)^2+1}\right) = f\left(-x, \frac{y}{(-x)^2+1}\right)$

If $a = 0$ one point
otherwise 2 points

A5 Let $f(x) = 0 \quad \forall x \in [0,\infty) = A$

$$\frac{x}{nx+n^2} \le \frac{x}{n^2} \text{ since } nx+n^2 \ge n^2 \text{ for } x \in A$$

$$\therefore S(x) = \sum_{n=1}^{\infty} \frac{x}{n(x+n)} \le x \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \text{ for } x \in A.$$

$\therefore S(x) \longrightarrow f(x)$ pointwise.

Now on $[0,b]$ $\quad \frac{x}{nx+n^2} \le \frac{b}{n^2}$

and $\sum_{n=1}^{\infty} \frac{b}{n^2} < \infty$ for $b$ fixed

Weierstrass M-test, the convergence is uniform on $[0,b]$.

b) By linearity of the integral,

$$\int_0^1 S_N(x)\,dx = \sum_{n=1}^{N} \left[\frac{1}{n}\int_0^1 dx - \int_0^1 \frac{1}{x+n}\,dx\right]$$

$$= \sum_{n=1}^{N} \left[\frac{1}{n} - \ln(x+n)\Big|_0^1\right]$$

$$= \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \left[\ln(1+n) - \ln(n)\right]$$

$$= \sum_{n=1}^{N} \frac{1}{n} - \left[\ln 2 - \underbrace{\ln 1}_{0} + \ln 3 - \ln 2 + \cdots + \ln(N+1) - \ln N\right]$$

$$= \left[\sum_{n=1}^{N} \frac{1}{N}\right] - \ln(N+1).$$

For $x \in [0,1]$, $S_N(x) \to \sum_{n=1}^{\infty} \frac{x}{n(x+n)}$

uniformly as $N \to \infty$ by a)

$\therefore \int_0^1 S_N(x)\,dx \underset{\uparrow \text{ by b)}}{=} \sum_{n=1}^{N} \frac{1}{n} - \ln(N+1)$

converges to $\int_0^1 \sum_{n=1}^{\infty} \frac{x}{n(x+n)}$

$\underset{\uparrow \text{ by uniform convergence}}{=} \sum_{n=1}^{\infty} \int_0^1 \frac{x}{n(x+n)}\,dx \le \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 x\,dx$

$\le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

$\left( n^2 + nx \ge n^2 \Rightarrow \frac{1}{n^2} \ge \frac{1}{n^2+nx} \text{ for } x \in [0,1] \right)$

$\boxed{B1}$ a) Suppose $E$ is not dense in $[0,1]$. Then $\bar{E} \ne [0,1]$, so $\exists\, y \in [0,1] / \bar{E}$

$\ni \exists\, \delta > 0 \ni E \cap ([0,1] \cap (y-\delta, y+\delta)) = \phi$

$m[0,1] = 1 \ge mE + \underbrace{m(\bar{E} \cap [0,1] \cap (y-\delta, y+\delta))}_{> \delta}$

$\therefore 1 - \delta \ge mE \Rightarrow \Leftarrow$

$\therefore E$ is dense in $[0,1]$

June 4)

$\boxed{3}$

B2 b) a msbl function $f$ is integrable over $E$ iff $f^+$ and $f^-$ are both integrable over $E \Leftrightarrow \int_E |f| < \infty$

Royden p 90

see Royden p 88

Let $f(x) = \log x$. Then $f'(x) = \frac{1}{x}$.

Now $\int_\varepsilon^1 f'(x)\,dx = \int_\varepsilon^1 \frac{1}{x}\,dx$

$= \log 1 - \log \varepsilon = -\log \varepsilon \to \infty$

as $\varepsilon \to 0$.

$\therefore \int_0^1 f'(x)\,dx = \infty$

$\therefore f'$ is not Leb msbl.

$\boxed{B3}$ ?

$\boxed{B4}$ Let $f_n \to f(x) \equiv 0$ a.e.

Since $|f_n| \le f_1$ a.e. and

$f_1$ is integrable, by the LCT,

$\lim_n \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} 0 = 0.$

Now $g$ nonnegative and $\int_E g = 0$

$\Rightarrow g = 0$ a.e. on $E$

contrapositive Suppose $\lim_n \int_{\mathbb{R}} f_n \ne 0$. Then $f_n \to 0$ a.e.

$\Rightarrow \lim_n \int_{\mathbb{R}} f_n = 0$ by LCT $\Rightarrow \Leftarrow$ $\therefore f_n \not\to 0$ a.e.

if $\lim \int_{\mathbb{R}} f_n \ne 0$.

empty interior. Then $\exists\, y \in$ interior $E$.

$\therefore \exists\, \delta > 0 \ni (y-\delta, y+\delta) \subseteq E$

$\therefore mE \geq m(y-\delta, y+\delta) = 2\delta > 0$

$\Rightarrow \Leftarrow$.

$\therefore E$ has empty interior

c) $[0,1] \cap \mathbb{Q}$ is dense in $[0,1]$

but $0 \leq m([0,1] \cap \mathbb{Q}) \leq m(\mathbb{Q}) = 0$

so a) has no converse.

$[0,1] \cap \overline{\mathbb{Q}}$ has empty interior

but $m\left[\underbrace{[0,1] \cap \overline{\mathbb{Q}}}_{\text{irrationals in } [0,1]}\right] = 1$

$\therefore$ converse of b) is false.

B2 a) $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x+h) - f(x)}{h}$

Since $f'$ exists on $\mathbb{R}$ $\therefore f$ is continuous on $\mathbb{R}$. $\therefore f$ is measurable on $\mathbb{R}$.

$g(x) = h \;\; \forall x$ is msbl. $\therefore$

Hence $f(x+h) - f(x)$ is msbl

also is $\dfrac{f(x+h) - f(x)}{h}$.

$\therefore \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = f'(x)$ is msbl.

take $h_n = \frac{1}{n}$ let $n \to \infty$.

$\therefore x^2 \cos\left(\frac{1}{x^2}\right)$ is diff on $\mathbb{R}\setminus\{0\}$

Jonet

now $\dfrac{f(x)-f(0)}{x-0} = x\cos\left(\frac{1}{x^2}\right)$

and $\displaystyle\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ since $\left|\cos\frac{1}{x^2}\right| \le 1$.

$\therefore f'(0)$ exists.

Thus $f'$ exists on $\mathbb{R}$.

Let $0, \dfrac{\sqrt{2}}{\sqrt{(2n+1)\pi}}, \dfrac{\sqrt{2}}{\sqrt{2n\pi}}, \dfrac{\sqrt{2}}{\sqrt{(2n-1)\pi}}, \ldots, \dfrac{\sqrt{2}}{\sqrt{2\pi}}, \dfrac{\sqrt{2}}{\sqrt{\pi}}, 1$

$= x_0, x_1, \ldots, x_{2n+1}, x_{2n+2}$ be a partition of $[0,1]$.

$\cos\frac{k\pi}{2} = 0$

$|\cos k\pi| = 1$

Then $\displaystyle\sum_{k=1}^{2n} |f(x_k) - f(x_{k-1})| \ge$

$2\left[\dfrac{2}{2n\pi} + \dfrac{2}{2(n-1)\pi} + \cdots + \dfrac{2}{4\pi} + \dfrac{2}{2\pi}\right]$

$= \dfrac{2}{\pi}\left[\dfrac{1}{1} + \dfrac{1}{2} + \cdots + \dfrac{1}{n}\right]$

$\to \infty$ as $n \to \infty$.

b) $F(x) = \int_0^x f(t)\,dt$ iff $F$ is absolutely continuous $\Rightarrow F$ is of bdd variation

($F$ is not of bdd var on $[0,1]$)

$\therefore F$ is not abs cont

$\therefore F(x)$ does not equal $\int_0^x f(t)\,dt$ $\forall x \in [0,1]$

B64 a) $z\bar{z} = |z|^2 = x^2+y^2$ (4)

$(x+iy)(x-iy)$

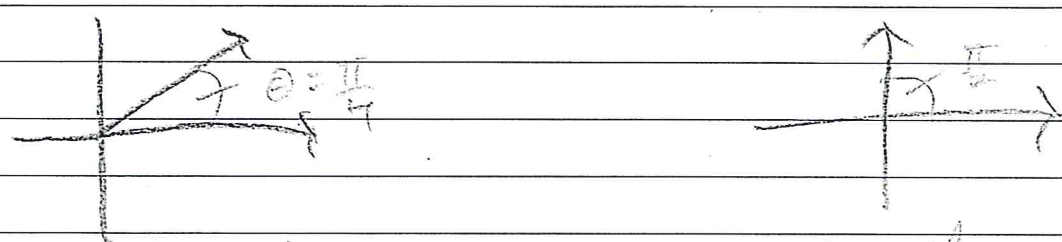
$\frac{\partial u}{\partial x} = 2x$   $\frac{\partial v}{\partial x} = 0$   $\frac{\partial v}{\partial y} = 0$

$\therefore$ Cauchy Riemann hold only at $z=0$
C-R eq'ns need to hold on some disk about $z=0$ for $f$ to be analytic at $z=0$.

$\boxed{F}$

b) $f$ analytic and $f'(z) \neq 0$ $\forall z \in \Omega$
$\Rightarrow$ $f$ is conformal on $\Omega$.

$f(z) = z^2$ is analytic on $\mathbb{C}$
$f'(z) = 2z = 0$ iff $z=0$




$z$ plane   $w$ plane

$W$ doubles the angle between any 2 rays with endpoints at $z=0$
$\therefore$ $W$ is not conformal

$\boxed{F}$ See Churchill 3rd edition p. 196-197

c) $f(z) = 1$ is entire

$f(z) = \{1\}$ is bounded

$\boxed{F}$

d) If $z = x+iy$   $|e^z| = |e^x e^{iy}| = e^x$

$\boxed{F}$ Take $M=2$  $x=0$  $|z| = \sqrt{0^2+y^2} = |y|$, for $|y|>R$
$|z|>R$ but $|e^z|<2$.

$f$ assumes no max in $G$.

But $G \cup \partial G$ is a closed set.

$f$ continuous on $G \Rightarrow \text{Re} f$ and $\text{Im} f$ are cont $\therefore |f| = \sqrt{(\text{Re} f)^2 + (\text{Im} f)^2}$ is a cont function. $\therefore |f|$ assumes a max on $G \cup \partial G = \overline{G}$. $\therefore$ statement is true.

$\boxed{B7}$ a) This must mean state a cor to a Cauchy thm. A cor to Cauchy's th in a disk is as follows.

Let $\Delta = \{ z \mid |z - z_0| < \rho < \infty \}$.

Suppose $f$ is analytic in $\Delta$ except at a finite # of points $\zeta_1, \ldots, \zeta_n$.

If $\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$ for $j = 1, \ldots, n$,

then $\int_\gamma f(z)\,dz = 0$ for every closed path in $\Delta \setminus \{\zeta_1, \ldots, \zeta_n\}$.

Here $z_0 = 0$, $\rho = 1$, $n = 1$, $\zeta_1 = z_0 = 0$.

b) $\int_\gamma \frac{\sin w}{(w-z)^2}\,dw = \frac{2\pi i}{1!} f'(z)$ where $f(w) = \sin w$

$= 2\pi i \cos z$ if $n(\gamma, z) = 1$

$0$ if $n(\gamma, z) = 0$

1.1.5.3 Ahlfors notes