Examination B

Analysis

January 1994

Instructions: All candidates should attempt 4 of the 5 problems in Part A. Those taking the 2-hour examination should work 4 of the 10 problems in Part B. Those taking the 3-hour examination should work 8 of the 10 problems in Part B. Clearly indicate which problems you wish to have scored.

PART A. Work 4 of the 5 problems.

A1. (a) Prove that if $\{a_n\}$ is an increasing sequence of real numbers bounded from above, then $\{a_n\}$ is convergent.

p 52-53 Gaughan

(b) Use the result in part (a) to prove that $\{a_n\}$ given by $a_1 = 1$, $a_n = \sqrt{2a_{n-1}}$ for $n \geq 2$, is convergent.

A2. (a) If $f(x) = \dfrac{3}{x-2}$, $3 \leq x \leq 6$, prove directly that f is uniformly continuous on $[3, 6]$.

p88 and p96 Gaughan

(b) Prove, using the definition of uniform continuity, that $f(x) = \dfrac{1}{x-1}$ is not uniformly continuous on $(1, 3]$. Can you quote a theorem which yields the same conclusion?

A3. Let

$$f(x) = \begin{cases} x^3 \sin 1/x & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Prove that f is continuous and differentiable for all x and that f' is continuous, but also that f' is not differentiable at $x = 0$.

Gaughan p 108

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A4. Let $f : R^2 \to R^1$ be given by

$$f(x, y) = 3x + y^2$$

(a) Prove that f is uniformly continuous on $D_1 = [0, 1] \times [0, 1]$ (an $\epsilon - \delta$ argument!)

(b) Prove that f is not uniformly continuous on $D_2 = [0, 1] \times [0, \infty)$.

A5. Let $f_k(x) = \dfrac{k^2 x}{1 + k^2 x^2}$.

(a) Find the pointwise $\lim_{k\to\infty} f_k(x)$ for $x \in R$.

(b) Show that $f_k(x)$ does not converge uniformly on any interval of the form $[0, c)$, $c > 0$, but does converge uniformly on $[1, \infty)$.

$$f_k'(x) = \frac{(1+k^2x^2)k^2 - k^2x \cdot 2k^2x}{(1+k^2x^2)^2}$$

$$= \frac{k^2 - k^4x^2}{(1+k^2x^2)^2} \overset{set}{=} 0$$

$$k^2(1-x^2) \overset{set}{=} 0$$

$$x = 1$$

$$f_k'(x) \leq 0 \quad \text{for } x \geq 1$$

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PART B. Work 4 of the 10 problems for the 2-hour examination. Work 8 of the 10 problems for the 3-hour examination. Clearly indicate which problems you wish to have scored.

B1. Let G be an open set and E be a measurable set in $\mathbb{R}$ with $m(E) = 0$. Prove

$$m\overline{G} = m(\overline{G \setminus E}),$$

where $\overline{A}$ is the closure of set A.

B2. Let f be a measurable function in a set E. Suppose that G is an open set and F is a closed set. Are

3.15 Schaum outline

$$E(f \in G) \equiv \{x \in E : f(x) \in G\} \text{ and } E(f \in F) \equiv \{x \in E : f(x) \in F\}$$

measurable? Justify your answers.

B3. Let E and E_i $(i = 1, 2, \ldots)$ be measurable sets in $\mathbb{R}$ and f be Lebesgue integrable over E. If $E_i \subset E$ for all i and $\lim_{i\to\infty} mE_i = mE < \infty$, show

$$\lim_{i\to\infty} \int_{E_i} f(x)\,dx = \int_E f(x)\,dx.$$

B4. Let f and f_n be measurable and non-negative functions in $[0, 1]$. Suppose

See Royden p92

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e. in } [0,1] \text{ and } \lim_{n\to\infty} \int_{[0,1]} f_n\,dx = \int_{[0,1]} f\,dx < \infty.$$

Show that if E is any measurable set in $[0, 1]$, then

$$\int_E f_n\,dx \to \int_E f\,dx.$$

B5. Let f be Lebesgue integrable in $[a, b]$. Set

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$$F(x) = \int_a^x f(t)\,dt, \text{ for } x \in [a, b].$$

Show that F is of bounded variation.

A1 a) Set $a = \sup a$

Let $\varepsilon > 0$. Then $\exists$ $n_0 \ni |a - a_{n_0}| < \varepsilon$

So $\forall\ n \geq n_0$ $\quad a - \varepsilon < a_{n_0} \leq a_n \leq a < a+$

$\therefore$ $\forall\ n \geq n_0$ $\quad |a_n - a_0| < \varepsilon$.

$\therefore$ $\lim a_n = a$.

b) Set $a_1 = 1$ then $a_2 = \sqrt{2} > 1$

Claim a_n is increasing use induction

True for $n = 2$. assume $a_K > a_{K-1}$ for $K = n-1$.

Then $a_n = \sqrt{2 a_{n-1}} = \sqrt{2}\ \sqrt{a_{n-1}}$

~~So~~ $a_n \geq a_{n-1}$ if $\sqrt{2} \geq \sqrt{a_{n-1}}$. or $2 \geq a_{n}$

Claim $a_K \leq 2$ $\forall K$.

True for a_1. assume true for $K = n-1$

$a_n = \sqrt{2\ a_{n-1}} = \sqrt{2}\ \sqrt{a_{n-1}} \leq \sqrt{2}\ \sqrt{2}$

$\therefore$ $a_K \leq 2$ $\forall K$

and a_K is increasing } by induction

so by a_1 $\{a_n\}$ is convergent.

A2 a) Let $\varepsilon > 0$. $\quad \frac{3}{4} \leq f(x) \leq 3 \quad \forall x \in [3, 6]$

Set $\delta = K\varepsilon$. If x_1 and $x_2 \in [3, 6]$

~~then~~ $|f(x_1) - f(x_2)| = \left| \dfrac{3}{x_1 - 2} - \dfrac{3}{x_2 - 2} \right|$

$= \left| \dfrac{3x_2 - 6 - 3x_1 + 6}{(x_1 - 2)(x_2 - 2)} \right| = \left| \dfrac{3(x_2 - x_1)}{(x_1 - 2)(x_2 - 2)} \right|$

$\leq \dfrac{3\delta}{1^2} = 3K\varepsilon$. Take $K =$

$x - 2 > 1$

$\frac{1}{x-2} \leq 1$

Continuous on compact set so some choice of δ will work

Since $f(x)$ is unbounded on $[1,3]$,
f is not uniformly continuous on $[1,3]$
hence not on $(1,3]$. Th 3.9 P96

Let $\varepsilon = 1$. Let $\delta > 0$.

Let $|x-y| = \frac{\delta}{2} < \delta$.

$$|f(x)-f(y)| = \left|\frac{1}{x-1} - \frac{1}{y-1}\right| = \left|\frac{y-1-x+1}{(x-1)(y-1)}\right|$$

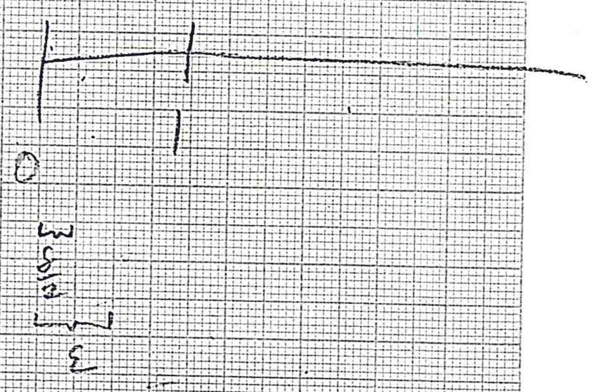
$$= \left|\frac{y-x}{(x-1)(y-1)}\right| =$$

Let $|x-1| < \sqrt{\frac{|x-y|}{k\varepsilon}}$, $|y-1| < \sqrt{\frac{|x-y|}{k\varepsilon}}$

then $\frac{|y-x|}{|x-1||y-1|} > \frac{|y-x|}{|y-x|} k\varepsilon = k\varepsilon$

Take $k = 1$. $\Rightarrow \varepsilon$

is this possible? yes



$|x-1| < \sqrt{\frac{\frac{\delta}{2}}{\varepsilon}}$

$|y-1| < \sqrt{\frac{\frac{\delta}{2}}{\varepsilon}}$

and $|x-y| < \frac{\delta}{2}$

A3 $\left|\sin \frac{1}{x}\right| \le 1$

$\therefore \lim_{x \to 0} x^3 \left|\sin \frac{1}{x}\right| = 0$

Since x^3 is continuous on $\mathbb{R}$

and $\sin \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$.

$x^3 \sin \frac{1}{x}$ is continuous on $\mathbb{R}$.

$\frac{f(x) - f(0)}{x - 0} = x^2 \sin\frac{1}{x} - 0$

$\therefore \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

$\therefore f'(0)$ exists

at $x \ne 0$ $f'(x) = 3x^2 \sin \frac{1}{x} + x^3 \cos \frac{1}{x} \left(-\frac{1}{x^2}\right)$

$= 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$

$\therefore f'(x) = \begin{cases} 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x} & x \ne 0 \\ 0, & x = 0 \end{cases}$

$x^2 \sin\frac{1}{x}$, x, and $\cos \frac{1}{x}$ are cont except

$\therefore f'(x)$ is cont on $\mathbb{R} \setminus \{0\}$,

Now $\lim_{x \to 0} f'(x) = 0$ since $\left|\sin \frac{1}{x}\right| \le 1$

and $\left|\cos \frac{1}{x}\right| \le 1$ $\forall x \in \mathbb{R} \setminus \{0\}$

$\therefore f'(x)$ is cont on $\mathbb{R}$.

But $\lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0} 3x \sin \frac{1}{x} - \cos \frac{1}{x}$

does not exist.

f is continuous $\therefore$ f is U.C.

Let $\varepsilon > 0$. $0 \le f(x,y) \le 4$ on D_1.

Let $\underline{x}_1, \underline{x}_2 \in D_1$ $\|\underline{x}_1 - \underline{x}_2\| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$< \delta = k\varepsilon$.

$|f(x_1,y_1) - f(x_2,y_2)| = |3(x_1 - x_2) + y_1^2 - y_2^2|$

$\le 3|x_1 - x_2| + |y_1 - y_2||y_1 + y_2|$

$\le 3k\varepsilon + k\varepsilon 2 = 5k\varepsilon$

take $k = \frac{1}{5}$.

b) Suppose f is U.C. on D_2.

Let $\varepsilon = 1$ and let $0 < \delta$.

Let $x_1 = x_2 = 0$ $y_1 \ne y_2$

$|\underline{x}_1 - \underline{x}_2| = |y_1 - y_2| = \frac{\delta}{2} < \delta$. $|y_1 + y_2| = \frac{\varepsilon}{\delta}$

$|f(\underline{x}_1) - f(\underline{x}_2)| = |y_1^2 - y_2^2| =$

$= |y_1 - y_2||y_1 + y_2| > \delta \frac{\varepsilon}{\delta} = \varepsilon \Rightarrow \Leftarrow$

eg $y_1 > y_2$. $y_1 - y_2 = \frac{\delta}{2}$, $y_1 = y_2 + \frac{\delta}{2}$

$y_1 + y_2 = \frac{\varepsilon}{\delta}$

$2y_2 = \frac{\varepsilon}{\delta} + \frac{\delta}{2}$, $y_2 = \frac{\varepsilon}{2\delta} + \frac{\delta}{4}$

$y_1 = \frac{\varepsilon}{2\delta} + \frac{3\delta}{4}$

A5)

$\lim_{n\to\infty} f_n(x) = \lim \dfrac{\frac{1}{n^2}}{\frac{1}{n^2}} \cdot \dfrac{n^2 x}{1+n^2 x^2} = \lim \dfrac{x}{\frac{1}{n^2} + x^2}$

$= \dfrac{x}{x^2} = \dfrac{1}{x} = f(x).$

b) Suppose $f_n \xrightarrow{u} f$ on $(0, c)$, $c > 0$.

Let $\varepsilon = 1$, for $x \in (0, c)$

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and let N be such that

$|f_n(x) - f(x)| < 1$ for all $n \geq N$.

$|f_n(x) - f(x)| = \left| \dfrac{n^2 x}{1+n^2 x^2} - \dfrac{1}{x} \right|$

$= \left| \dfrac{-1}{(1+n^2x^2)x} \right| < 1 \quad \forall n \geq N$

In particular $\dfrac{1}{(1+N^2x^2)x} < 1 \quad \forall x \in (0, c)$

$\Rightarrow\Leftarrow$ if x is close enough to 0,

$\dfrac{1}{(1+N^2x^2)x} > 1.$

On $[1, \infty)$, $|f_n(x) - f(x)| =$

$\dfrac{1}{(1+n^2x^2)x} \leq \dfrac{1}{1+n^2}$.

since $1+n^2 \leq (1+n^2x^2)x \quad \forall x \in [1, \infty)$.

$$|f_n(x) - f(x)| = \frac{1}{(1+n^2x^2)x} \leq \frac{1}{1+n^2} < \varepsilon$$

if $1+n^2 > \frac{1}{\varepsilon} \Rightarrow n^2 > \frac{1}{\varepsilon} - 1 > \frac{1}{\varepsilon}$

Take N an integer $> \frac{1}{\sqrt{\varepsilon}}$

Then $n \geq N \Rightarrow n^2 > \frac{1}{\varepsilon}$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x \in [1, \infty)$

$\therefore$ $f_n(x)$ does converge uniformly on $[1, \infty)$.

B1

$G = (G \cap E^c) \cup (G \cap E) = G \setminus E \cup (G \cap E)$ ← zero measure

$\overline{A \cup B} = \overline{A} \cup \overline{B}$

$\therefore \overline{G} = \overline{G \setminus E} \cup \overline{G \cap E}$.

$G \setminus E \subseteq G \quad \therefore m(\overline{G \setminus E}) \leq m(\overline{G})$

So $\overline{G \setminus E} \subseteq \overline{G}$

$m\,G/E + 0 = m(G) = m(G \setminus E) \quad \therefore m(G) = m(G \setminus E)$

$0 = m(\phi) = m(Q) \qquad \overline{\phi} = \phi, \quad \overline{Q} = \mathbb{R}$

$\overline{G} = \overline{G \setminus E} \cup \overline{(G \cap E)} = \overline{G \cap E} \cup (\overline{G \cap E} \cap \overline{G \setminus E}$

$\therefore m(\overline{G}) = m\,\overline{G \setminus E} + m[\overline{G \cap E} \cap (\overline{G \setminus E})^c]$

$\forall x \in (\overline{G \setminus E})^c = (\overline{G \cap E^c})^c$

$G \cap E^c \subseteq \overline{G \cap E^c}$

$\therefore (\overline{G \cap E^c})^c \subseteq (G \cap E^c)^c$

$G \cap E^c \subseteq E^c \quad \therefore E \subseteq (G \cap E^c)^c$

Stuck

$f^{-1}(G) = \{x \in E \mid f(x) \in G\}$

$f^{-1}(F) = \{x \in E \mid f(x) \in F\}$

Let $G = \bigcup_{k=1}^{\infty} I_k$, $I_k = (a_k, b_k)$ the disjoint component intervals.

$$E(f(x) \in G) = \{x \in E : f(x) \in \bigcup_{k=1}^{\infty} (a_k, b_k)\}$$

$$= \bigcup_{k=1}^{\infty} \{x \in E : f(x) \in (a_k, b_k)\} = \bigcup_{k=1}^{\infty} E(I_k).$$

f is a function and I_k are disjoint $\therefore f(x) \in G \iff f(x) \in$ exactly one I_k say I_{k_0}.

$$= \bigcup_{k=1}^{\infty} \Big[\underbrace{E(f(x) > a_k)}_{\in \mathfrak{M}} \cap \underbrace{E(f(x) < b_k)}_{\in \mathfrak{M}}\Big]$$

$\in \mathfrak{M}$ since $\mathfrak{M}$ is a σ field.

ii) $(f^{-1}(F))^c = f^{-1}(\underbrace{F^c}_{\text{open}}) \in \mathfrak{M}$ by i

$\therefore (f^{-1}(F^c))^c = f^{-1}(F) \in \mathfrak{M}$

B3) $|E(X_n)| = |E(X_n)|$ [illegible] $\le$ [illegible]

[illegible] $E X_n = E X$ [illegible]

[illegible] DCT [illegible]

[illegible]

Since E_n is monotonically [illegible]

B4

$E X_n \to$ [illegible]

By the dominated convergence theorem (DCT)

[illegible]

B5

[illegible]

On D_n [illegible]

[illegible]

[illegible]

[illegible]