

Contents

Preface	vii
Chapter 1. Introduction: Crosscutting Issues	1
1.1. Ergodic Systems	1
1.2. Entropy	1
1.3. Discrete and Continuous Structures	1
1.4. Approximation	2
1.5. Bounding Complexity	2
Chapter 2. Formulating Probability	3
2.1. Axioms for Probability	3
2.1.1. The Prehistory of Probability	3
2.1.2. The Kolmogorov Axiomatization	5
2.2. Reasoning With Probability	9
2.2.1. The Primacy of Probability	9
2.2.2. Probability Logic	14
2.3. Axiomatizing and Computing Conditionality	18
2.3.1. Independence and Bayesian Networks	18
2.3.2. Nonmonotonic Logic and Conditionality	23
2.4. Adapted Spaces and Distributions	25
2.5. Approximate Measure Logic	27
2.6. Continuous First-Order Logic and Metric Structures	30
2.6.1. Continuous First-Order Logic	30
2.6.2. Some Metric Structures	34
2.7. Continuous First Order Logic as a Generalization	36
Chapter 3. Random Sequences	39
3.1. Normal Sequences	39
3.1.1. Popper and Randomness	39
3.1.2. The Problem of Bases, and Normal Numbers	41
3.1.3. Computability of Normal Numbers	45
3.2. Martin-Löf Randomness	47
3.2.1. Kolmogorov Complexity	47
3.2.2. Martin-Löf Characterization by measures	50
3.2.3. Characterization by Games	52
3.3. Other Notions of Algorithmic Randomness	56
3.3.1. n -Randomness	56
3.3.2. Shnorr Randomness	57
3.3.3. Stochasticity	59
3.4. Effective Dimension	61

3.4.1. Effective Hausdorff Dimension	61
3.4.2. Packing Dimension	65
3.4.3. Box (a.k.a. “Box Counting,” or “Minkowski” Dimension)	66
3.5. An Example: Brownian Motion	67
Chapter 4. Nondeterminism and Randomized Computation	71
4.1. Nondeterminism	71
4.1.1. Nondeterministic Machines	71
4.1.2. NP and the Polynomial Hierarchy	72
4.1.3. Descriptive Complexity	73
4.2. Randomized Turing Machines	74
4.2.1. Complexity Classes Defined by Randomization	74
4.2.2. Effective Completeness for Continuous First Order Logic	80
4.2.3. Pseudorandom Generators and Derandomization	83
4.3. Interactive Proofs	84
4.3.1. Interactive Proofs as Games	84
4.3.2. Interactive Proofs as Proofs	89
4.4. Word Problems	91
4.4.1. Groups with Solvable Word Problem	91
4.4.2. Groups with Generically Solvable Word Problem	93
4.5. Generic and Coarse Computability	96
4.5.1. Generic Computability	96
4.5.2. Coarse Computability	99
4.5.3. Relationships between Generic and Coarse	101
4.5.4. Beginnings of Generically and Coarsely Computable Structure Theory	103
Chapter 5. Pseudofinite Objects and 0–1 Laws	105
5.1. 0–1 Laws	105
5.1.1. Theories of Random Graphs	105
5.1.2. The Almost Sure Theory	108
5.2. Fraïssé Limits	113
5.2.1. Fraïssé’s Theorem	113
5.2.2. Ehrenfeucht-Fraïssé Games and 0–1 Laws	116
5.3. Model Theory of Pseudofinite Structures	118
5.3.1. Pseudofinite Fields	118
5.3.2. Pseudofinite Groups	124
5.3.3. Classes of Finite Structures and Pseudofinite Structures	126
5.4. The Lovasz Local Lemma	130
5.4.1. The Local Lemma and the probabilistic Method	130
5.4.2. The Computable Lovász Local Lemma	133
Chapter 6. Random Structures	137
6.1. Graphons	137
6.1.1. Defining Graphons	137
6.1.2. Invariant Measures	142
6.1.3. Entropy Methods for Graphons and Invariant Measures	147
6.2. Keisler Randomizations	152
6.2.1. The Idea of Randomization Structures	152

6.2.2. The Randomization Theory	156
6.3. Algorithmically Random Structures	159
6.3.1. Algorithmic Randomness and Selection Randomness	159
6.3.2. Defining Algorithmically Random Structures	160
6.3.3. Glasner-Weiss Measures and Their Random Structures	164
6.3.4. Haar Measure and Haar Compatible Measure	168
6.4. Invariant Random Subgroups	170
6.4.1. Defining Invariant Random Subgroups	170
6.4.2. Classifying Invariant Random Subgroups	173
6.5. Probabilistic Boolean Networks	178
6.5.1. An Application: Gene Regulatory Networks	178
6.5.2. Defining Probabilistic Boolean Networks	179
6.5.3. Some Problems on Probabilistic Boolean Networks	180
Chapter 7. Learning and Independence	183
7.1. The Learning Problem	183
7.2. Learning c.e. Sets to Equality and Almost Equality	184
7.3. Probably Approximately Correct Learning and Vapnik–Chervonenkis Dimension	189
7.3.1. Allowing Randomization and Errors	189
7.3.2. Classification by Regression	191
7.3.3. Vapnik-Chervonenkis Dimension	194
7.3.4. The Index Set for PAC Learnable Classes	202
7.3.5. PAC reducibility	204
7.4. NIP Theories	205
7.4.1. The Independence Property	205
7.4.2. Examples of NIP Theories	209
7.4.3. Learning in NIP Theories	210
7.4.4. Learning in Other Theories	211
Chapter 8. Dynamics	213
8.1. Julia Sets Considered Computationally	213
8.2. Polish Spaces, Orbit Equivalences, and Borel Reducibility	216
8.2.1. Polish Spaces and Orbit Equivalences	216
8.2.2. Borel Cardinality and Reducibility	218
8.2.3. The Glimm-Effros Dichotomy	220
8.3. Classification and Ergodic Theory	222
8.3.1. Amenable Groups	222
8.3.2. Ergodicity	228
8.3.3. Turbulence	233
8.4. Incomparable Equivalence Relations	234
8.4.1. Zimmer Superrigidity	234
8.4.2. Incomparable Relations Under Measure Reducibility	239
8.4.3. Computably Enumerable Equivalence Relations	241
8.5. Universal Minimal Flows	244
8.6. Realizations of Invariant and Stationary Measures	246
8.7. Stochastic and Combinatorial Regularity	249
8.7.1. Szemerédi Regularity Lemma	249
8.7.2. The Furstenberg Correspondence Principle	256

8.7.3. Markov Chains	258
Chapter 9. Some Problems	261
9.1. Problems on Formulating Probability	261
9.2. Problems on Random Sequences	261
9.3. Problems on Randomized Computation	262
9.4. Problems on Pseudofinite Objects	263
9.5. Problems on Random Structures	263
9.6. Problems on Learning	264
9.7. Problems on Dynamics	264
Appendix A. Preliminaries in Logic	265
A.1. Classical First-Order Logic	265
A.2. Proofs	267
A.3. Recursion and Computability	267
A.4. Sets	267
A.5. Extensions of Classical First-Order Logic	267
Bibliography	269
Index	287