GENERICALLY COMPUTABLE STRUCTURES

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Abstract. We define notions of generically and coarsely computable and c.e. relations, structures, and functions. We examine this notion in several specific families of structures, including graphs, abelian groups, equivalence relations and injection structures.

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For example, a binary relation R on ω is generically computable if there is a partial computable function $\phi : \omega \times \omega \to \{0,1\}$ such that $\phi = \chi_R$ on the domain of ϕ and there is a c.e. set of asymptotic density one such that $A \times A$ is a subset of the domain of ϕ; the set A and the relation R are said to be *faithful* if, whenever $a \in A$ and aRb, then $b \in A$. The relation R is said to be generically Σ_1 if there is a c.e. set A such that A is asympotically dense and (A, R) is a Σ_1 substructure of (ω, R). It is shown that every equivalence structure $\mathcal{E}$ has a generically computable copy. However, $\mathcal{E}$ has a generically Σ_1 copy if and only if it has an infinite faithful substructure with a computable copy. Furthermore, $\mathcal{E}$ has a faithful generically computable copy if and only if it has an infinite faithful substructure with a computable copy.

Experts in computability and complexity can show that many problems are hard to solve, or even unsolvable. Thus many results in computable structure theory tend to depend sensitively on the construction of adversarial (and frequently *ad hoc*) examples. As a well-known example, a standard construction of a finitely presented group with unsolvable word problem [10] involves not just getting the right example of a group; the particular words within this group on which it is difficult to decide equality to the identity are very special words (and are even called by this term in some expositions). In another well-known example from complexity theory, the simplex algorithm is known to have exponential complexity in the worst case, but empirically runs in much shorter time on practically all inputs.

It would be worthwhile to distinguish which results in computable structure theory depend on a "special" (and potentially extremely rare) input, and which are less sensitive. To do this job in the context of word problems on groups, Kapovich, Myasnikov, Schupp, and Shpilrain proposed using notions of asymptotic density to state whether a partial recursive function could solve "almost all" instances of a problem [8].

Jockusch and Schupp [6] generalized this approach to the broader context of computability theory in the following way.

Definition 0.1. Let $S \subseteq \mathbb{N}$.

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(1) The density of S up to n, denoted by $\rho_n(S)$, is given by

$$\frac{|S \cap \{0, 1, 2, \ldots, n\}|}{n+1}.$$

(2) The asymptotic density of S, denoted by $\rho(S)$, is given by $\lim_{n\to\infty} \rho_n(S)$.

A set A is said to be *generically computable* if and only if there is a partial computable function ϕ such that ϕ agrees with χ_A throughout the domain of ϕ, and such that the domain of ϕ has asymptotic density 1. A set A is said to be *coarsely computable* if and only if there is a *total* computable function ϕ that agrees with χ_A on a set of asymptotic density 1. We will need the following result from [6]

Theorem 0.2 (Jockusch-Schupp)**.** *There is a generically computable set which is not coarsely computable and there is a coarsely computable set which is not generically computable.*

The study of generically and coarsely computable sets and some related notions has led to an interesting program of research in recent years; see [5] for a partial survey. The purpose of the present paper is to examine notions of generically and coarsely computable functions, relations, and structures and to present some results for equivalence structures and isomorphisms.

Given a structure $\mathcal{A}$ with universe ω, and finitely many functions $\{f_i : i \in I\}$, each f_i of arity p_i and relations $\{R_j : j \in J\}$, each R_i of arity r_j, we want to consider what it means to say that $\mathcal{A}$ is generically computable, or "nearly computable" in some other notion related to density. The idea is that $\mathcal{A}$ is generically computable if there is a substructure $\mathcal{D}$ with universe a c.e. set D of asymptotic density one which is computable in the following sense: There exist partial computable functions $\{\phi_i : i \in I\}$ and $\{\psi_j : j \in J\}$ such that ϕ_i agrees with f_i on D^{p_i} and ψ_j agrees with the characteristic function of R_j on D^{r_j}. Similarly $\mathcal{A}$ is coarsely computable if there is a computable structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with universe D is a substructure of both $\mathcal{A}$ and of $\mathcal{E}$ and all relations and functions agree on D. A more interesting variaton requires that $\mathcal{D}$ is a Σ_1-elementary submodel of $\mathcal{A}$, more generally a Σ_n-elementary submodel. That is, if we are saying that $\mathcal{A}$ is "nearly computable" when it has a dense substructure $\mathcal{D}$ which is computable (c.e.), then the substructure should be similar to $\mathcal{A}$ by some standard.

To be precise, recall that $\mathcal{D}$ is an Σ_n-elementary) substructure of $\mathcal{A}$ provided that, for any Σ_1 formula $varphi(x_1, \ldots, x_n)$ and any elements $a_1, \ldots, a_n \in D$,

$$\mathcal{A} \models \varphi(a_1, \ldots, a_n) \iff \mathcal{D} \models \varphi(a_1, \ldots, a_n).$$

We will say that the structure $\mathcal{A}$ is *generically* Σ_n if there is an asymptotically dense set D such that

(a) $\mathcal{D}$ is a Σ_n-elementary substructure of $\mathcal{A}$;
(b) there exist partial computable functions $\{\phi_i : i \in I\}$ such that ϕ_i agrees with f_i on D^{p_i};
(c) each R_j restricted to D^{r_j} is a c.e. relation.

We remark that generically computable is the same as generically Σ_0, since $\mathcal{B}$ is a submodel of $\mathcal{A}$ if and only if it preserves all quantifier-free formulas.

Notions of coarsely Σ_n structures will also be defined.

Here are some examples.

Example 0.3. Consider structures of the form $\mathcal{A} = (\omega, A)$, where A is a unary relation, that is to say A is a set. Suppose that the set A is genericaly computable and let ϕ be a partial computable function such that $D = Dom(\phi)$ is a dense c.e. set and, for $x \in D$, $\phi(x) = \chi_A(x)$. Then the substructure $\mathcal{D} = (D, A \cap D)$ is a c.e. substructure of $\mathcal{A}$ since ϕ is total on the set D and therefore $\mathcal{A}$ is a generically computable structure. On the other hand, suppose that $\mathcal{A}$ has a substructure $\mathcal{D} = D, A \cap D)$ where D is a c.e. dense set and $A \cap D$ is a computable set. Then there is a partial computable function ϕ with domain D such that ϕ agrees with χ_A on D, and it follows that the set A is generically computable.

Next suppose that A is coarsely computable and let $f : \omega \to \{0,1\}$ be a total computable function, let $E = \{x : f(x) = 1\}$, and let D be a dense set such that f agrees with χ_A on D. Let $\mathcal{E} = (\omega, E)$. This is in fact a computable structure. Then $A \cap D = E \cap D$, so that $\mathcal{D} = (D, A \cap D)$ is a substructure of both $\mathcal{A}$ and $\mathcal{E}$. Thus $\mathcal{A}$ is a coarsely computable structure. On the other hand, suppose that there is a dense set D and a computable structure $\mathcal{E} = (\omega, E)$ such that $\mathcal{E}$ agrees with $\mathcal{A}$ on the set D, that is to say $A \cap D = E \cap D$. Then χ_A agrees with the total computable function $f = \chi_E$ on the dense set D, so that A is coarsely computable.

Example 0.4. Let $\mathcal{A} = (A, E)$ be a countable directed graph consisting of infinitely many finite chains of distinct lengths. Let $C(\mathcal{A})$ be the set of lengths of the chains. The structure $\mathcal{A}$ is c.e. if A is a c.e. set and E is a c.e. relation. For a c.e. structure $\mathcal{A}$, $C(\mathcal{A})$ will be a Σ^0_2 set. Then $\mathcal{A}$ is generically computable if there is an asymptotically dense set c.e. D such that E is computable on D.

We will also be interested in the question of whether a structure $\mathcal{A}$ has a generically computable copy, more generally a generically Σ_n copy. In this example, any such structure $\mathcal{A}$ will have a generically computable copy $\mathcal{B}$. Build the generically computable copy as follows: Let $D = \{d_0 < d_1 < \cdots\}$ be an asymptotically dense, co-infinite computable set and put edges from d_{2n} to d_{2n+1} for each n. Then use $\omega \setminus D$ to fill out the needed $c_n - 1$ vertices at the front of each chain to obtain a copy of $\mathcal{A}$.

Suppose now that $\mathcal{D}$ is a Σ_1-elementary substructure of such a graph $\mathcal{A}$. Then for each $a \in A$, the chain containing a must be included in $\mathcal{D}$; let us say that $\mathcal{D}$ is a *faithful* substructure when this happens. For example, if a is in the chain $a_0EaEa_2Ea_3$, then $\mathcal{A} \models (\exists x)xEa)$. Thus $\mathcal{D} \models (\exists x)xEa)$, and therefore $a_0 \in D$. Similarly $\mathcal{A} \models (\exists y)(\exists z)aEy \wedge yEz)$, and therefore a_2 and a_3 must be in $\mathcal{D}$. Thus a structure $\mathcal{A}$ will be generically Σ_1 if there is an asymptotically dense set c.e. set D such that $\mathcal{D} = (D, E)$ is a faithful substructure of $\mathcal{A}$ and $E \cap (D \times D)$ is a c.e. relation of $\mathcal{A}$. Then the structure $\mathcal{A}$ will have a generically Σ_1 copy if and only if there exists $C \subseteq C(\mathcal{A})$ and a c.e. structure $\mathcal{D}$ with $C(\mathcal{D}) = C$.

Finally, suppose that $\mathcal{D} = (D, E)$ is a Σ_2 elementary substructure of $\mathcal{A}$. This will imply that $C(\mathcal{D}) = C(\mathcal{A})$ and therefore $D = A$. It follows that $\mathcal{A}$ is generically Σ_2 if and only if $\mathcal{A}$ is a c.e. structure. Thus a structure $\mathcal{A}$ has a generically Σ_2 copy if and only if it has a c.e. copy.

Example 0.5. Let $sA = (A, f)$ be a countable directed graph consisting of cycles of length $c_n \geq 3$ with distinct c_n, where the edge relation aEb is given by $f(a) = b$. Let $C(\mathcal{A}) = \{c_n : n \in \omega\}$. If $\mathcal{A}$ is a c.e. structure, then $C(\mathcal{A})$ will be a c.e. set. Conversely, for each such set c.e. C of distinct natural numbers ≥ 3, there is a computable graph with $C(\mathcal{A}) = C$. A substructure $\mathcal{D}$ will consist of an arbitrary subset of the cycles, so that $C(\mathcal{D}) \subseteq C(\mathcal{A})$. It follows that $\mathcal{A}$ is generically computable if and only if there is an asymptotically dense c.e. set D such that $C(D,E)$ is a c.e. set and the edge relation is computable on D.

If $\mathcal{A}$ has a generically computable copy $\mathcal{B}$, then that copy has a substructure $\mathcal{D}$ with $C(\mathcal{D})$ an infinite c.e. set, so that $C(\mathcal{A})$ has an infinite c.e. subset. Next suppose that $C(\mathcal{A})$ has an infinite c.e. subset C. Then we can define a computable structure $\mathcal{D}$ with $C(\mathcal{D}) = C$ on a dense co-infiniite set D and then fill out the rest of $\mathcal{D}$ so that $C(\mathcal{D}) = C(\mathcal{A})$. Thus a structure $\mathcal{A}$ of this type will have a generically computable copy if and ony if $C(\mathcal{A})$ has an infinite c.e. subset.

A Σ_1 elementary substructure $\mathcal{D}$ must have $C(\mathcal{D}) = C(\mathcal{A})$, since for $c \in C(\mathcal{A})$, $\mathcal{A}$ satisfies the Σ_1 sentence $(\exists x)[f^{(c)}(x) = x \wedge (\forall i < c) f^{(i)}(c) \neq c]$. Under the conditions above, the only Σ_1-elementary substructure of $\mathcal{A}$ is $\mathcal{A}$ itself. Thus we see that $\mathcal{A}$ is generically Σ_1 if and only if it $\mathcal{A}$ is a computable structure.

This example may also be viewed as an injection structure. We will examine injection structures later in more detail. Next we give an example from the study of equivalence structures. We will also return to this topic in more detail.

Example 0.6. Let $\mathcal{A} = (A, E)$ where E is an equivalence relation with infinitely many classes of size k for k in the infinite set $C = C(\mathcal{A})$, and also infinitely many infinite classes. If $\mathcal{A}$ is c.e., then $C(\mathcal{A})$ is a Σ^0_2 set and, conversely, for any Σ^0_2 set C there is such a computable structure $\mathcal{A}$ with $C(\mathcal{A}) = C$. A substructure $\mathcal{D} = (D, E)$ will have, for each equivalence class of $\mathcal{A}$, a (possibly empty) subclass. As in Example 0.4, every such structure $\mathcal{A}$ will have a generically computable copy.

Suppose now that $\mathcal{D}$ is a Σ_1-elementary substructure of such an equivalence structure $\mathcal{A}$. Then for each $a \in D$, there are two cases to consider. If the equivalence class $[a]_{\mathcal{A}}$ is finite, then $[a]_{\mathcal{D}}$ must equal $[a]_{\mathcal{A}}$. If $[a]_{\mathcal{A}}$ is infinite, then $[a]_{\mathcal{D}}$ must also be infinite. We note that there need not be any infinite classes at all in $\mathcal{D}$. Then one can construct a generically Σ_1 copy of $\mathcal{B}$ of $\mathcal{A}$ by letting $\mathcal{B}$ have infinitely many infinite classes which is on an asympotically dense computable set and arbitrarily filling in the rest of $\mathcal{B}$ to match the class sizes from $\mathcal{A}$.

Finally, suppose that $\mathcal{D}$ is a Σ_2-substructure of $\mathcal{A}$. Then $\mathcal{D}$ must have infinitely many classes of size k for each $k \in C(\mathcal{A})$. For example, if $2 \in C(\mathcal{A})$, then $\mathcal{A} \models (\exists x, y)[xEy \wedge x \neq y \wedge (\forall z)(xEz \rightarrow (z = x \vee z = y)]$, so that $2 \in C(\mathcal{D})$. Thus if $\mathcal{A}$ is generically $\Sigma - 2$, then $C(\mathcal{A})$ must be a Σ^0_2 set. It follows that $\mathcal{A}$ has a generically Σ_2 copy if and only if $C(\mathcal{A})$ is a Σ^0_2 set.

Example 0.7. Fix a prime p and suppose that $\mathcal{A} = \oplus_{n \in C}\mathbb{Z}(p^n)$ for some infinite set C. If $\mathcal{A}$ is computable, then C is a Σ^0_2 set and furthermore C has an s_1 function; details are given below. Conversely, for any Σ^0_2 set C with an s_1-function, there is such a computable structure $\mathcal{A}$ isomorphic to $\oplus_{n \in C}\mathbb{Z}(p^n)$.

Any such structure will have a generically computable copy. Let $\mathcal{A} = \oplus_{i<\omega}\langle a_i \rangle$, where $o(a_i) = p^{n_i}$. Then consider the subgroup $\mathcal{B} = \oplus_{i<\omega}\langle p^{n_i - 1} a_i \rangle$, which is isomorphic to $\oplus_{i<\omega}\mathbb{Z}(p)$. We observe that $\mathcal{B}$ is not a Σ_1-elementary subgroup, since for each $n_i > 1$, $p^{n_i - 1}$ has height $n_i - 1$ in $\mathcal{A}$ but has height 1 in $\mathcal{B}$. $\mathcal{B}$ has a computable copy, and we can construct a generically computable copy of $\mathcal{A}$ with the corresponding subgroup on an asymptotically dense set.

Suppose now that $\mathcal{D}$ is a Σ_1-elementary subgroup of $\mathcal{A}$. Then $\chi(\mathcal{B}) \subseteq \chi(\mathcal{A})$. If $\mathcal{A}$ is generically Σ_1, then $\chi(\mathcal{A})$ has a Σ^0_2 subset which possesses an s_1-function. Thus if $\mathcal{A}$ has a generically Σ_1 copy, then C must have a Σ^0_2 subset with an s_1-function.

Finally, suppose that $\mathcal{B}$ is a Σ_2-elementary subgroup of $\mathcal{A}$. Then $\chi(\mathcal{B}) = \chi(\mathcal{A})$. To see this, let $n \in C$. Then in $\mathcal{A}$, there exists an a such that $o(a) = p^n$ and $\langle a \rangle$ is a pure subgroup of $\mathcal{A}$. But this is a Σ^0_2 sentence, and therefore $\mathcal{B}$ also has such an element a. If $\{n_i : i < \omega\}$ is distinct, then in fact $\mathcal{B} = \mathcal{A}$.

These notions prove quite interesting for certain families of structures. We will examine in some detail the notions of generically computable and coarsely computable structures, and the variations described above for injection structures, equivalence structures, and also Abelian p-groups.

The outline of this paper is as follows. Section 1 contains background on asymptotic density, and gives the generalizations of generic and coarse computability to structures. We show that a set A has asymptotic density δ if and only if the set $A \times A$ has density δ^2 in $\omega \times \omega$. We show that there is a computable dense set $C \subset \omega \times \omega$ such that for any infinite computably enumerable set A, the product $A \times A$ is not a subset of C.

Section 2 presents results on generically computable and generically Σ_n injection structures. We show that an injection structure $\mathcal{A}$ has a generically computable copy if and only if it has an infinite substructure which is isomorphic to a computable injection structure, and that $\mathcal{A}$ has a generically Σ_1 copy if and only if it has a computable copy.

Section 3 presents results for equivalence structures. We extend the lemma from [1] to show that any computably enumerable equivalence relation on a computably enumerable set, with no infinite equivalence classes and with unbounded character, possesses an s_1-function (a technical auxilliary that is frequently useful in this area, which we will define). We present the unexpected result that every equivalence structure $\mathcal{A}$ has a generically computable copy. We show that $\mathcal{A}$ has a generically Σ_1 copy if and only if it has an infinite substructure which is isomorphic to a c.e. structure, and that $\mathcal{A}$ has a generically Σ_2 copy if and only if it has a c.e. copy.

Section 4 presents results related to coarse computability.

In Section 5, we discuss current and future work on this project.

1. Generically and coarsely computable sets and structures

In this section, we provide some background on the notions of generically computable and coarsely computable sets. We define more general notions of generically Σ_n, structures, and also coarsely computable and coarsely Σ_n structures. Then we examine these notions when applied to injection structures and to equivalence structures.

The asymptotic density of a set $A \subseteq \omega$ is defined as follows.

Definition 1.1.
- The *upper asymptotic density* of A is $\limsup_n \frac{|(A \cap n)|}{n}$;
- The *lower asymptotic density* of A is $\liminf_n \frac{|(A \cap n)|}{n}$
- The *asymptotic density* of A is $\lim_n \frac{|(A \cap n)|}{n}$, if this exists.

It is easy to see that A has asymptotic density δ if and only if A has both upper and lower density δ; A has density 1 if and only if it has upper density 1 and A has density 0 if and only if it has lower density 1

In [6], Jockusch and Schupp give the following definitions.

Definition 1.2. Let $S \subseteq \omega$.

(1) We say that S is *generically computable* if there is a partial computable function $\Phi : \omega \to 2$ such that $\Phi = \chi_S$ on the domain of Φ, and such that the domain of Φ has asymptotic density 1.
(2) We say that S is *coarsely computable* if there is a computable set T such that $S \triangle T$ has asymptotic density 0, that is, there is a computable function $f : \omega \to \{0, 1\}$ which agrees with χ_S on a set of density one.

It was shown in [6] that there is a coarsely computable computably enumerable set which is not generically computable, and a generically computable computably enumerable set which is not coarsely computable.

The following observations will be useful.n Note that the set A has *upper density* 1 if and only if there is a sequence $n_0 < n_1 < \cdots$ such that $\lim_i \frac{|A \cap n_i|}{n_i} = 1$.

Lemma 1.3. *If A is a computably enumerable set with upper density one, then A has a computable subset with upper density one.*

Proof. Suppose that A is a computably enumerable set with upper density 1. Define computable sequences $n_0, n_1, n_2, \ldots$ and $s_0, s_1, s_2, \ldots$ as follows. Let $n_0 = s_0 = 0$. Let s_1 be the least s such that, for some $n < s$, we have $|n \cap A_s| \geq \frac{1}{2}n$, and let n_1 be the least such n. Given n_k and s_k, let s_{k+1} be the least s such that, for some n with $n_k < n < s$, we have $|(n - n_k) \cap A_s| \geq \frac{2^{k+1}-1}{2^{k+1}}(n - n_k)$, and let n_{k+1} be the least such n. The computable dense set $B \subseteq A$ is defined so that, for each i, if $n_k \leq i < n_{k+1}$, then $i \in B \iff i \in A_{n_{k+1}}$. It follows from the construction that, for each k, the density of B in $\{i : i > n_k\}$ is at least $\frac{2^k-1}{2^k}$, so that B has upper density 1. □

In order to study binary relations and the corresponding structures, we need to look at notions such as generic computability for such relations.

Lemma 1.4. *Let $A \subset \omega$. Then A has asymptotic density δ if and only if $A \times A$ has asymptotic density δ^2 in $\omega \times \omega$. In particular, A is asymptotically dense in ω iff $A \times A$ is asymptotically dense in $\omega \times \omega$. More generally, if A has asymptotic density δ_A and B has asymptotic density δ_B, then $A \times B$ has asymptotic density $\delta_A \cdot \delta_B$.*

Proof. Let $\delta_A(n) = \frac{|A \cap n|}{n}$ and let $\delta(n) = \frac{|(A \times A) \cap (n \times n)|}{n^2}$. Since $(A \times A) \cap (n \times n) = (A \cap n) \times (A \cap n)$, it follows that $|(A \times A) \cap n \times n| = |A \cap n|^2$ and hence $\delta(n) = \delta_A(n)^2$. If $\lim_n \delta_A(n) = \delta$ exists, then $\lim_n \delta(n) = \lim_n \delta_n(A)^2 = \delta^2$. Conversely, if $\lim_n \delta(n) = L = \delta^2$ exists, then $\lim_n \delta_A(n) = \lim_n \sqrt{\delta_n(A)} = \sqrt{L} = \delta$.

For the second part, let $\delta_A(n) = |A \cap n|/n$ and $\delta_B(n) = |B \cap n|/n$ and suppose that $\delta_A = \lim_n \delta_A(n)$ and $\delta_B = \lim_n \delta_n(B)$ both exist. Then $\delta(n) = |(A \times B) \cap (n \times n)| = \delta_A(n) \times \delta_B(n)$ so $lim_n \delta(n) = \delta_A \cdot \delta_B$ is the asymptotic density of $A \times B$. □

A similar result holds for the density of A^r in ω^r. On the other hand, we have the following.

Theorem 1.5. *There is a computable dense $C \subset \omega \times \omega$ such that for any infinite computably enumerable set $A \subset \omega$, the product $A \times A$ is not a subset of C.*

Proof. Define C as follows. For any pair (a, b) with $max\{a, b\} = m$, proceed as follows. For each $e < m$, look for the first element $n > 2^e$ which has come in by stage m; call this n_e if it exists. Then put $(a, b) \in C$, unless either $a = n_e$ or $b = n_e$

for some $e < m$. If W_e is infinite, then it contains some element $n_e > 2^e$ which is the first to come into W_e at some stage s_e, and then there will be another $n \in W_e$ which is greater than s_e but (n_e, n) will not be in C. The set C is dense since there are at most i elements less than 2^i of the form n_e for any $e < i$ so that C contains at least $(2^i - i)^2$ elements out of the 2^{2i} possible pairs up to 2^i. □

In consideration of Lemma 1.4 and Theorem 1.5, our definition of a generically computable structure with a binary relation calls for a dense set D in the domain so that the relation is computable on $D \times D$ rather than for a dense set in $\omega \times \omega$ where the relation is computable. The most natural notion seems to be require that the substructure with domain D resembles the given structure $\mathcal{A}$ by agreeing on certain first-order formulas, existential formulas in particular. We recall the notion of an elementary substructure.

Definition 1.6. A substructure $\mathcal{B}$ of the structure $\mathcal{A}$ is said to be an *elementary* substructure ($\mathcal{B} \prec \mathcal{A}$) if for any $b_1, \dots, b_n \in \mathcal{B}$, and any formula ϕ, $\mathcal{A} \models \phi(b_1, \dots, b_n) \iff \mathcal{A} \models \phi(b_1, \dots, b_n)$.

The substructure $\mathcal{B}$ is said to be a Σ_n-elementary substructure ($\mathcal{B} \prec_n \mathcal{A}$) if for any $b_1, \dots, b_n \in \mathcal{B}$, and any Σ_n formula ϕ, $\mathcal{A} \models \phi(b_1, \dots, b_n) \iff \mathcal{A} \models \phi(b_1, \dots, b_n)$.

Definition 1.7. For any structure $\mathcal{A}$:

(1) A substructure $\mathcal{B}$ of $\mathcal{A}$, with universe B, is a *computable substructure* if the set B is c.e and each function and relation is computable on B, that is, for any k-ary function f and any k-ary relation R, both $f \restriction B^k$ and $\chi_R \restriction B^k$ are the restrictions to b^k of partial computable functions.
(2) A substructure $\mathcal{B}$ of $\mathcal{A}$, with universe B, is a *computably enumerable (c.e.) structure* if the set B is c.e., each relation is c.e. and the graph of each function is c.e. (so that the function is partial computable but also total on B).
(3) $\mathcal{A}$ is *generically computable* if there is a substructure $\mathcal{D}$ with universe a c.e. set D of asymptotic density one such that the substructure $\mathcal{D}$ with universe D is a computable substructure.
(4) $\mathcal{A}$ is *generically* Σ_n if there is a dense c.e. set D such that the substructure $\mathcal{D}$ with universe D is a c.e. substructure and also a Σ_n-elementary substructure of $\mathcal{A}$.

For $n > 0$, any generically Σ_{n+1} structure is generically Σ_n. For structures with functions but no relations, this also holds for $n = 0$. However, a c.e. substructure might not be computable, so a structure $\mathcal{A}$ with relations which is generically Σ_1 is not necessarily generically computable.

In the following sections, we will study specific families of structures, that is injection structures, equivalence structures, and abelian p-groups, and also consider

2. Injection Structures

Definition 2.1. An *injection structure* $\mathcal{A}$ is a set A together with a one-to-one function $f : A \to A$. $\mathcal{A}$ is computable (respectively c.e.) if $A \subseteq \omega$ is computable (resp. c.e.) and f is the restriction of a partial computable function to A. The *orbit* $\mathcal{O}_f(a)$ under f is

$$\mathcal{O}_f(a) = \{x : (\exists n \in \omega)[x = f^{(n)}(a) \ \vee \ a = f^{(n)}(x)]\}.$$

Orbits are either finite or infinite. Infinite orbits may be of type $\mathbb{Z}$ where $\mathcal{O}_f(a) = \{\dots, f^{-2}(a), f^{-1}(a), a, f(a), f^2(a), \dots\}$ or of type ω, where for some b not in the range of f, $\mathcal{O}_f(a) = \{b, f(b, f^{(2}(b), \dots\}$. The *character* $\chi(\mathcal{A})$ of $\mathcal{A}$ is

$$\chi(\mathcal{A}) = \{(k, n) \in (\omega \setminus \{0\}) \times (\omega \setminus \{0\}) : \mathcal{A}\text{has at least } n \text{ orbits of size } k\}.$$

Definition 2.2. A set $K \subseteq (\omega \setminus \{0\}) \times (\omega \setminus \{0\})$ is said to be a *character* if, for all k and n, $(k, n+1) \in K$ implies $(k, n) \in K$.

It is easy to see that K is a character if and only if $K = \chi(\mathcal{A})$ for some injection structure $\mathcal{A}$.

Computable and c.e. injection structures were investigated by the authors together with A. Morozov [2] and by Cenzer, Harizanov and Remmel [**?**], where the following are shown.

Lemma 2.3. *For any c.e. injection structure $\mathcal{A}$,*

(1) $\{(a, k) : a \in Ran(f^{(k)})\}$ *is a c.e. set;*
(2) $\{(a, k) : card(\mathcal{O}_{f(a)}) \geq k\}$ *is a c.e. set;*
(3) $\{a : \mathcal{O}_f(a) \textit{is infinite}\}$ *is the intersection of a* Π^0_1 *set with* A*;*
(4) $\{a : \mathcal{O}_f(a) \textit{has type } \mathbb{Z}\}$ *is a* Π^0_2 *set;*
(5) $\{a : \mathcal{O}_f(a) \textit{has type } \omega\}$ *is a* Σ^0_2 *set;*
(6) $\chi(\mathcal{A})$ *is a c.e. set.*

Proposition 2.4. *For any c.e. character K, there is a computable injection structure $\mathcal{A} = (\omega, f)$ with character K and any specified finite or countably infinite number of orbits of types ω and $\mathbb{Z}$. Furthermore the range of f is computable and $\{a : \mathcal{O}_f(a)$ is finite$\}$ is computable.*

The following lemma is needed.

Lemma 2.5. *Any c.e. injection structure is isomorphic to a computable injection structure.*

Proof. Given an infinite c.e. set A and a partial computable function f which is an injection on A, let $A = \{\varphi(0), \varphi(1), \dots\} = Ran(\varphi)$, where φ is a computable injection from ω onto A and let $g(n) = \varphi^{-1}(f(\varphi(n)))$. Then φ is an isomorphism from the computable injection structure $\mathcal{E} = \omega, g)$ to $\mathcal{A} = (A, f)$, since $\varphi(g(n)) = f(\varphi(n))$. □

Proposition 2.6. *For any injection structure $\mathcal{A} = (\omega, f)$, $sA = (\omega, f)$ has a generically computable copy if and only if $\mathcal{A}$ has an infinite substructure $\mathcal{B}$ which is isomorphic to a c.e. injection structure.*

Proof. Suppose first that $\mathcal{A} = (\omega, f)$ has a generically computable copy $\mathcal{C} = (\omega, g)$ and let $H : \mathcal{C} \to \mathcal{A}$ be an isomorphism. Now by definition there is a dense c.e. set D such that $\mathcal{D}$ is a c.e. substructure of $\mathcal{C}$; D must be infinite since it is dense. Then the image $\mathcal{B} = (H(D), f)$ is an infinite substructure of $\mathcal{A}$ which is isomorphic to $\mathcal{D}$.

Next suppose that $\mathcal{A} = (\omega, f)$ has an infinite substructure $\mathcal{B} = (B, f)$ which is isomorphic to a c.e. injection structure with universe ω. We may assume that B is coinfinite, since otherwise $\mathcal{A}$ is a computable structure and hence also generically computable. Now let D be a coinfinite dense computable set and use the enumeration of D to convert this into a structure $\mathcal{D} = (D, g)$ which is isomorphic to $\mathcal{B}$. This means there is a set isomorphism $F : B \to D$ such that, for all $b \in B$, $F(f(b)) = g(F(b))$. Since B and D are both coinfinite, we may extend F to a permutation of ω mapping $\omega \setminus B$ to $\omega \setminus D$. Then we may extend D to a generically computable injection structure $\mathcal{C} = (\omega, g)$ by defining $g(x)$ to be $F(f(F^{-1}(x)))$, so that F will be an isomorphism between $\mathcal{A}$ and $\mathcal{C}$. □

Note that in the proof of Proposition 2.6, we obtain a generically computable copy with a dense *computable* substructure $\mathcal{D}$.

Proposition 2.7. *An injection structure $\mathcal{A} = (\omega, f)$ has a generically computable copy if and only at least one of the following holds:*

(1) *$\mathcal{A}$ has an infinite orbit;*
(2) *$\chi(\mathcal{A})$ has an infinite c.e. subset.*

Proof. Suppose that $\mathcal{A}$ has a generically computable copy. Then by Proposition 2.6, $\mathcal{A}$ has infinite substructure $\mathcal{D}$ which is isomorphic to a c.e. injection structure $\mathcal{C}$. There are two cases.

Case I: If $\mathcal{C}$ has an infinite orbit, then $\mathcal{D}$ has an infinite orbit $\mathcal{O}_f(a)$, and that orbit is also infinite in $\mathcal{A}$.

Case II: If $\mathcal{C}$ has no infinite orbits, then $\chi)\mathcal{C})$ is an infinite c.e. set and $\chi(\mathcal{C}) = \chi(\mathcal{D})$. But any finite orbit in $\mathcal{D}$ is also an orbit in $\mathcal{A}$ and it follows that $\chi(\mathcal{D})$ is an infinite c.e. subset of $\mathcal{A}$.

For the other direction, suppose first that $\mathcal{A}$ has an infinite orbit $\mathcal{O}_f(a)$. Then by Proposition 2.4, there is a computable injection structure consisting of exactly one orbit of the same type as $\mathcal{O}_f(a)$. Thus the orbit $\mathcal{O}_f(a)$ composes an infinite substructure of $\mathcal{A}$ which is isomorphic to a c.e. injection structure. It follows from Proposition 2.6 that $\mathcal{A}$ has a generically computable copy.

Next suppose that $\mathcal{A}$ has no infinite orbits and that $\chi(\mathcal{A})$ has an infinite c.e. subset K. Then again by Proposition 2.4, there is a computable structure with character K. So it again follows from Proposition 2.6 that $\mathcal{A}$ has a generically computable copy. □

Next we consider generically Σ_1 injection structures. First we characterize when $\mathcal{B}$ is a Σ_1 substructure of an injection structure $\mathcal{A}$.

Proposition 2.8. *$\mathcal{B}$ is a Σ_1-elementary substructure of the injection structure $\mathcal{A} = (\omega, f)$ if and only if*

(i) *For all $b \in B$, the orbit of b in $\mathcal{B}$ equals the orbit of b in $\mathcal{A}$;*
(ii) *$\chi(\mathcal{A}) = \chi(\mathcal{B})$.*
(iii) *If $\mathcal{A}$ has an infinite orbit, then either $\chi(\mathcal{B})$ is unbounded or $\mathcal{B}$ has an infinite orbit.*

Proof. Suppose that $\mathcal{B}$ is a Σ_1-elementary substructure of $\mathcal{A} = (\omega, f)$. Certainly finite orbits and orbits of type ω are equal in $\mathcal{B}$ and in $\mathcal{A}$, since $\mathcal{B}$ is closed under the function f. Since $\mathcal{B} \prec_1 \mathcal{A}$, if $\mathcal{A} \models (\exists x)\ f(x) = b$, then $\mathcal{B} \models (\exists x)\ f(x) = b$, so that $\mathcal{B}$ is also closed under f^{-1} and this preserves the orbits ot type $\mathbb{Z}$. Since finite orbits are preserved, $\chi(\mathcal{B}) \subseteq \chi(\mathcal{B})$. The other inclusion follows from $\mathcal{B} \prec_1 \mathcal{A}$. That is, let $\phi_k(x)$ be the formula $f^{(k)}(x) = x\ \wedge (\forall j < k) f^{(j)}(x) \neq x$. Then

$$(k, n) \in \chi(\mathcal{A})\mathcal{A} \models (\exists x_0, \ldots, x_{n-1})[(\forall i < n)\phi_k(x_i) \wedge (\forall i < j < n)(\forall t < k) f^{(t)}(x_i) \neq x_j.$$

Since this is a Σ_1 formula, it follows that $(k, n) \in \chi(\mathcal{A})$ implies $(k, n) \in \chi(\mathcal{B})$.

Finally, suppose that $\mathcal{A}$ has an infinite orbit. Then for each k, then $\mathcal{A}$ satisfies the sentence:

$$\psi_k : (\exists x)(\forall i \leq k) f^{(i)}(x) \neq x.$$

Then $\mathcal{B} \models \psi_k$ as well. Now suppose that $\chi(\mathcal{A})$ was bounded below k. Then there is some b such that $\forall i \leq k) f^{(i)}(x) \neq x$ and therefore $\mathcal{O}_f(b)$ must be infinite.

For the other direction, suppose that $\mathcal{B}$ satisfies the three conditions. Let $b_1, \ldots, b_m \in B$ and consider an arbitrary Σ_1 formula

$$\varphi(b_1, \ldots, b_m) : (\exists x_1, \ldots, x_n)\theta(b_1, \ldots, b_m, x_1, \ldots, x_n),$$

where θ is quantifier-free. By distributing disjunctions in the usual way, we may assume without loss of generality that θ is a conjunction of equalities and inequalities among some finite set of images $f^{(s)}(b_i)$ and $f^{(t)}(x_j)$. Since f is an injection, any equality of the form $f^{(s)}(b_i) = f^{(t)}(x_j)$ lets us eliminate x_j from the formula. Now suppose that $\theta(b_1, \ldots, b_m, a_1, \ldots, a_n)$. If any a_j is in the orbit of some b_i, then by (i), $a_j \in B$ and may be eliminated from θ. Thus the formula reduces to some $\theta'(a_1, \ldots, a_n)$. The equalities may be reduced to the form $a_h = f^{(t)}(a_j)$. If we have $a_j = f^{(t)}(a_j)$, then the orbit of a_j has type t. Since a_j is not in $\mathcal{O}_f(b_i)$ for any i, and $\chi(\mathcal{A}) = \chi(\mathcal{B})$, there must exist $c \in B$ with order type t not in any of $\mathcal{O}_f(b_i)$ and that $c = c_j$ may be substituted for a_j. For the other equalities of the form $a_h = f^{(t)}(a_j)$, we need an orbit in $\mathcal{B}$ of size $\geq t$ and such an orbit exists by (iii). Thus we can find c_h and c_j in B with $c_h = f^{(t)}(c_j)$. In the end we have $c_1, \ldots, c_n \in B$ so that $\mathcal{B} \models \theta(b_1, \ldots, b_m, c_1, \ldots, c_n)$ and therefore $\mathcal{B} \models \varphi(b_1, \ldots, b_m)$. □

For injection structures, the generically Σ_1 structures have a simple characterization.

Theorem 2.9. *The following are equivalent for any injection structure $\mathcal{A} = (\omega, f)$.*

(a) *$\mathcal{A}$ has a generically Σ_1 copy;*
(b) *$\chi(\mathcal{A})$ is a c.e. set;*
(c) *$\mathcal{A}$ has a computable copy;*
(d) *$\mathcal{A}$ has a generically Σ_2 copy;*

Proof. The key is to show that (a) implies (b). Suppose that $\mathcal{A}$ has a generically Σ_1 copy $\mathcal{E} = (\omega, g)$ and let D be a dense c.e. set such that $\mathcal{D} = (D, g) \prec_1 \mathcal{E}$ and $\mathcal{D}$ is a c.e. structure. Then $\chi(\mathcal{D})$ is a c.e. set and, by Proposition 2.8, $\chi(\mathcal{D}) = \chi(\mathcal{E})$. Since $\mathcal{A}$ is isomorphic to $\mathcal{E}$, it follows that $\chi(\mathcal{A})$ is a c.e. set. Proposition 2.4 shows that (b) implies (c). The implication from (c) to (d) is easy, since any computable structure is generically Σ_n, for any n. Any generically Σ_{n+1} structure is generically Σ_n, so (d) implies (a). □

3. Generically Σ_n Equivalence Structures

An equivalence structure $\mathcal{A} = (A, R)$ is simply a set with an equivalence relation R on A.

Definition 3.1. For any equivalence structure $\mathcal{A} = (A, R)$, the *character* $\chi(\mathcal{A})$ of $\mathcal{A}$ is $\{(k, n) : \mathcal{A}$ has at least n equivalence classes of size $k\}$.

We will sometimes just refer to the character of R when the set A is implicit.
Equivalence structures also have a character, defined as follows.

Definition 3.2. The *character* $\chi(\mathcal{A})$ of an equivalence structure $\mathcal{A} = (A, E)$ is

$$\chi(\mathcal{A}) = \{(k, n) \in (\omega \backslash \{0\}) \times (\omega \backslash \{0\}) : \mathcal{A} \text{has at least } n \text{ equivalence classes of size } k\}.$$

Let $Fin(\mathcal{A}) = \{a : [a] \text{ is finite}\}$ and $Inf^{\mathcal{A}} = \{a : [a] \text{ is infinite}\}$. As for injection structures, it is easy to see that K is a character if and only if $K = \chi(\mathcal{A})$ for some injection structure $\mathcal{A}$.

Computable and c.e. equivalence structures were studied by A. Morozov and the authors in [1] and by Cenzer, Harizanov and Remmel [3], where the following were shown.

Lemma 3.3. *For any c.e. equivalence structure $\mathcal{A}$,*

(1) $\{(a, k) : |[a]| \geq k\}$ *is a c.e. set;*
(2) $\{(a, k) : |[a]| = k\}$ *is a difference of c.e. sets;*
(3) $Inf^{\mathcal{A}}$ *is a* Π^0_2 *set;*
(4) $\chi(\mathcal{A})$ *is a* Σ^0_2 *set.*

Proposition 3.4. *Let K be a Σ^0_2 character. Then*

(1) *There is a computable equivalence structure $\mathcal{A} = (\omega, E)$ with character K and with infinitely many infinite equivalence classes. Furthermore $Inf^{\mathcal{A}}$ is a Π^0_1 set.*
(2) *For any finite $m \geq 1$, there is a c.e. equivalence structure $\mathcal{A} = (\omega, E)$ with character K and with exactly m many infinite equivalence classes.*

Definition 3.5. The function $f : \omega^2 \to \omega$ is said to be an s_1*-function* if the following hold:

(1) For every i and s, $f(i, s) \leq f(i, s + 1)$.
(2) For every i, the limit $m_i = \lim_{s\to\infty} f(i, s)$ exists.
(3) For every i, $m_i < m_{i+1}$.

The character K is said to *possess* the s_1-function f if it has an equivalence class of size m_i for each i. Here are some useful results about the characters of equivalence relations.

The first is a slight improvement of Lemma 2.1(c) of [3].

Lemma 3.6. *For any computably enumerable equivalence relation R on a computably enumerable set A, the character $\chi(R)$ is a Σ^0_2 set.*

Proof. The Lemma from [3] applies to a structure with universe ω. If R is only defined on the computably enumerable set A, just let $S(x, y) \iff (R(x, y) \lor x = y)$. This adds some classes of size 1 to the character, so that $\chi(S)$ is Σ^0_2 if and only if $\chi(S)$ is Σ^0_2. □

The next lemma is part of Lemma 2.8 of [1].

Lemma 3.7. *For any Σ^0_2 character K which possesses a computable s_1-function, there is a computable equivalence structure $\mathcal{E}$ with character K and no infinite equivalence classes.*

The next result is an improvement of Lemma 2.6 of [1]. It follows from the previous Lemma 3.7 that it also holds for structures restricted to a computably enumerable universe.

Lemma 3.8. *Let $\mathcal{A} = (\omega, E)$ be a computably enumerable equivalence structure with no infinite equivalence classes and an unbounded character. Then there is a computable s_1-function f such that $\mathcal{A}$ contains an equivalence class of size m_i for all i, where $m_i = lim_s f(i,s)$.*

Proof. Let E^p be the p^{th} stage in the enumeration of E, so that $E = \cup_p E^p$. We will define a uniformly computable family a_i^s for $i \leq s$ in such a way that $a_i = lim_s a_i^s$ exists. We will also define a computable sequence p_s, and let

$$f(i,s) = |\{a \leq p_s : aE^{p_s}a_i^s\}| .$$

Hence, we will have

$$m_i = \lim_s \ (|\{a \leq p_s : aE^{p_s}a_i\}| = |[a_i]|) .$$

At *stage* 0, we have $p_0 = 0$ and $a_0^0 = 0$, so $f(0,0) = 1$. In fact, a_0^s will equal 0 for all s.

After stage s, we have p_s and $a_0^s, \dots, a_s^s$ with $f(i,s)$ as above such that

$$f(0,s) < f(1,s) < \cdots < f(s,s).$$

At *stage* $s+1$, we define the least $p > p_s$ and the lexicographically least sequence $b_0, \dots, b_{s+1}$ such that for all $i \leq s$,

$$f(i,s) \leq |\{a \leq p : aE^p b_i\}| < |\{a \leq p : aE^p b_{i+1}\}| ,$$

as follows. Let $b_0 = a_0 = 0$. Furthermore, $b_i = a_i^{s+1}$ whenever there do not exist a pair a, j with $j \leq i$, $aE^p a_j^s$ and $p_s < a \leq p$. Then we let $a_i^{s+1} = b_i$ for each i and let $p_{s+1} = p$.

To see that such p exists, let m be the largest such that $[a_j^s] = \{a \leq p_s : aE^{p_s}a_j^s\}$ for all $j \leq m$, and let $b_i = a_i^s$ for all $i \leq m$. Then use the fact that $\chi(\mathcal{A})$ is unbounded to find $b_{m+1}, \dots, b_{s+1}$ with

$$|[a_m^s]| < |[b_{m+1}]| < |[b_{m+2}]| < \cdots < |[b_{s+1}]| ,$$

and take p large enough so that $[b_i] = \{a \leq p : aE^p b_i\}$.

Finally, we verify that $a_i = lim_s a_i^s$ exists for each i. Since there is no $j < 0$, it follows from the construction that $a_0^s = 0$ for all s. Given t such that $a_i = lim_s a_i^s$ has converged by stage t for all $i \leq k$, let $r \geq t$ be large enough so that

$$[a_i] = \{a < p_r : aE^{p_r}a_i\}$$

for all $i \leq k$. (This uses the fact that there are no infinite classes.) It follows from the construction that $a_{i+1}^s = a_{i+1}^r$ for all $s > r$. □

Proposition 3.9. *If $\mathcal{E}$ is a computably enumerable equivalence structure with no infinite equivalence classes, then $\mathcal{E}$ is isomorphic to a computable structure.*

Proof. By Lemma 3.6, $\mathcal{E}$ has a Σ_2^0 character, and by Lemma 3.8, this character possesses a computable s_1-function. Then by Lemma 3.7, there is a computable structure with the same character and no infinite equivalence classes, and hence isomorphic to $\mathcal{E}$. □

This last result also holds for a computably enumerable structure $\mathcal{E} = (A, E)$ where A is a computably enumerable set.

Now we consider equivalence structures in the context of generically computability and the variants thereof.

Theorem 3.10. *If an equivalence structure $\mathcal{E} = (\omega, E)$ is generically computable, then there is some infinite computable $Y \subseteq \omega$ such that the restriction of E to $Y \times Y$ is computable.*

Proof. Let Φ be the partial computable function and let A be an asymptotically dense computably enumerable set, given by the definition above. Then, by Lemma 1.3, A has a computable subset Y with upper density 1 (and thus infinite) with $Y \times Y \subseteq Dom(\Phi)$. Then $\chi_E = \Phi$ on the computable set Y. $\square$

Note that the set Y from the proof of Theorem **??** may not presere the equivalence classes of of $\mathcal{E}$.

Example 3.11. Let $K = \{(1, k) : k \in C\}$ where C has no infinite Σ^0_2 subset. Also take an immune set B. Then define $\mathcal{E}$ so that B is one infinite class, and $\omega \setminus B$ has character K. Then, while $\mathcal{E}$ itself need not be computable, $\mathcal{E}$ has a generically computable copy, where the infinite class is a dense computable set. Now let Y be an infinite computable subset of ω. Since B is immune, $Y \setminus B$ is infinite, so that Y has infinitely many elements with finite equivalence classes. If (Y, E) has a computable copy, then this copy has a Σ^0_2 character which is a subset of C. Thus at least (Y, E) is not a faithful substructure.

The following result was unexpected.

Proposition 3.12. *Every equivalence structure $\mathcal{E} = (\omega, E)$ has a generically computable copy.*

Proof. The proof is by cases. If $\chi(\mathcal{E})$ is bounded or if $\mathcal{E}$ has infinitely many infinite classes, then the result follows from Theorem 3.14. If $\mathcal{E}$ has an infinite equivalence class, B be such a class and let D be a computable dense set. Then we can define a generically computable copy $\mathcal{A} = (\omega, R)$ of $\mathcal{E}$ so that D is an infinite equivalence class and $(\omega \setminus B, E)$ is isomorphic to $(\omega \setminus B, R)$.

Next, suppose that $\mathcal{E}$ has no infinite equivalence class and $\chi(\mathcal{E})$ is unbounded. Then there must be infinitely many different k such that $\mathcal{E}$ has an equivalence class of size k. Choose one such class B_k for each k, and let $B \subseteq \omega$ consist of exactly one element from each class B_k. Then the substructure (B, E) consists of infinitely many classes of size one; notice that $\omega \setminus B$ is infinite. Now let $D \subset \omega$ be a computable, co-infinite set of asymptotic density one, and let let f be a permutation of ω mapping D onto B, and thus mapping $\omega \setminus D$ onto $\omega \setminus B$. Then we may define a generically computable copy (ω, R) of $\mathcal{E}$ by letting $xRy \iff f(x)Ef(y)$. Then R is computable on the computable dense set D, since for $x, y \in D$, we have $xRy \iff x = y$. $\square$

For equivalence structures, the generically Σ_1 structures have a nice characterization. Note that any substructure $\mathcal{B}$ of an equivalence structure $\mathcal{A}$ is also an equivalence structure, since the definitions of reflexive, symmetric, and transitive are all universal.

Proposition 3.13. *$\mathcal{B}$ is a Σ_1-elementary substructure of the equivalence structure $\mathcal{A} = (\omega, E)$ if and only if*

1. *For all $b \in B$, if $[b]_A$ is finite, then $[b]_a = [b]_B$ and if $[b]_A$ is infinite, then $[b]_B$ is infinite;*
2. *For any $k, n \in \omega$, if $\mathcal{A}$ has at least n classes of size $\geq k$, then $\mathcal{B}$ has at least n classes of size $\geq k$.*

Proof. One direction is immediate from the definition of Σ_1-elementary. For example, if $[b]_A = \{a, b, c\}$, then

$$\mathcal{A} \models (\exists x)(\exists y)[b \neq x \wedge b \neq y \wedge x \neq y \wedge bEx \wedge bEy \wedge xEy],$$

Then $\mathcal{B}$ must also satisfy this formula, so that $[b]_B$ has at least 3 members and therefore $[b]_B = \{a, b, c\} = [b]_A$.

For the other direction, suppose that $\mathcal{B}$ satisfies the two conditions. Let $b_1, \ldots, b_m \in B$ and consider an arbitrary Σ_1 formula

$$\varphi(b_1, \ldots, b_m) : (\exists x_1, \ldots, x_n)\theta(b_1, \ldots, b_m, x_1, \ldots, x_n),$$

where θ is quantifier-free.By distributing disjunctions in the usual way, we may assume without loss of generality that θ describes a partition of the set $\{b_1, \ldots, b_m, x_1, \ldots, x_n\}$. Suppose now that $\mathcal{A} \models \theta(a_1, \ldots, a_n, b_1, \ldots, b_m)$ and consider a particular equivalence class $\{a_{i_1}, \ldots, a_{i_k}, b_{j_1}, \ldots, b_{j_\ell}\}$. If necessary, simply the formula so that no two elements are equal. Let $b = b_{j_1}$. There are three cases to consider.

(1) Suppose that $[b]_A$ is finite. Then by condition (1), $[b]_B = [b]_A$, so that $a_{i_1}, \ldots, a_{i_k}$ belong to $[b]_B$.
(2) Suppose that $[b]_A$ is infinite. Then by condition (2). $[b]_B$ is also infinite, so that there are $b_{i_1}, \ldots, b_{i_k}$ so that the set $\{a_{i_1}, \ldots, a_{i_k}, b_{j_1}, \ldots, b_{j_\ell}\}$ may be replaced by the set $\{b_{i_1}, \ldots, b_{i_k}, b_{j_1}, \ldots, b_{j_\ell}\}$ in the partition described by θ.
(3) Finally, suppose $\ell = 0$ so that the equivalence class is just $\{a_{i_1}, \ldots, a_{i_k}$. Then by condition (3), there is an equivalence class in $\mathcal{B}$ with at least k elements which is disjoint from $\{b_1, \ldots, b_m\}$ and we may choose $\{b_{i_1}, \ldots, b_{i_k}$ from such a class.

It follows that elements $b'_1, \ldots, b'_n$ may be chosen so that $\mathcal{B} \models \theta(b'_1, \ldots,_n, b]_1, \ldots, b_m)$ and therefore $\mathcal{B} \models \varphi(b_1, \ldots, b_m)$. □

Theorem 3.14. *An equivalence structure $\mathcal{A} = (\omega, E)$ has a generically Σ_1 copy if and only if at least one of the following holds:*

(a) *$\chi(\mathcal{A})$ is bounded;*
(b) *$\chi(\mathcal{A})$ has a Σ^0_2 subset K with an s_1-function;*
(c) *$\mathcal{A}$ has an infinite class and $\chi(\mathcal{A})$ has a Σ^0_2 set subset K;*
(d) *$\mathcal{A}$ has infinitely many infinite classes.*

Proof. If $\mathcal{A}$ has a generically computable copy, then it has a Σ_1-elementary substructure which is isomorphic to a c.e. structure. Thus one of the cases (a,b,c,d) must hold.

(a): If $\chi(\mathcal{A})$ is bounded, then $\mathcal{A}$ has a computable copy.

In cases (b) and (c), we will assume that $\chi(\mathcal{A})$ is unbounded and show that there is $\mathcal{B} \prec_1 \mathcal{A}$ which is isomorphic to a c.e. structure $\mathcal{D}$, then build a copy $\mathcal{C}$ of $\mathcal{A}$ with a dense c.e. substructure $\mathcal{D}$ and fill out the rest of $\mathcal{C}$ to make it isomorphic to $\mathcal{A}$.

(b): In this case, $\mathcal{A}$ has a substructure $\mathcal{B}$ with unbounded character K and no infinite classes, which will therefore be a Σ_1-elementary substructure. By Lemma 3.7, there is a computable structure with character K isomorphic to $\mathcal{B}$ and we may define a structure $\mathcal{D} = (D, R)$ on a computable dense set D with $|\omega \setminus D| = |\omega \setminus B|$. Let ψ be a set isomorphism from $\omega \setminus D$ to $\omega \setminus B$ and extend R to $\omega \setminus D$ by letting $xRy \iff \psi(x)E\psi(y)$. Then ψ will extend the isomorphism of $\mathcal{D}$ and $\mathcal{B}$ to an isomorphism of $\mathcal{A}$ and (ω, R). The structure (ω, R) is generically Σ_1 since it has a dense c.e. Σ_1-elementary substructure $\mathcal{D}$.

(c): This is similar to part (b) except that $\mathcal{B}$ now has an infinite class as well. It is important to note that we define a c.e. structure $\mathcal{D} = (D, R)$ on a *computable* dense set D, though the relation R is c.e. and may not be computable.

(d): In this case, the substructure $\mathcal{B}$ consisting of the infinite classes will be a Σ_1-elementary substructure and we proceed as in (b) to define a c.e. dense structure $\mathcal{D}$ with infinitely many infinite classes and extend this to a generically c.e. structure (ω, R) which is isomorphic to $\mathcal{A}$. $\square$

We observe that the argument above also proves that $\mathcal{A}$ is generically Σ_1 if and only if it has a substructure $\mathcal{B}$ which is isomorphic to a c.e. structure.

The generically Σ_2 equivalence structures have a simple characterization.

Proposition 3.15. *If $\mathcal{B}$ is a Σ_2-elementary substructure of the equivalence structure $\mathcal{A} = (\omega, E)$, then $\chi(\mathcal{A} = \chi(\mathcal{B}$ and if $\mathcal{A}$ has an infinite classes, then either $\mathcal{B}$ has an infinite class or $\chi(\mathcal{B})$ is unbounded.*

Proof. Let $\mathcal{B} = (B, E)$ be a Σ_2-elementary submodel of an equivalence structure $\mathcal{A} = (\omega, E)$ be an equivalence structure. Then $\chi(\mathcal{B}) = \chi(\mathcal{A})$. This is because there is a Σ_2 formula $\psi_{n,k}$ which states that $(n, k) \in \chi(\mathcal{A})$.

Next suppose that $\mathcal{A}$ has an infinite equivalence class but $\mathcal{B}$ does not have an infinite class. Then for each k, $\mathcal{A}$ has a class of size at least k, that is, $\mathcal{A} \models \psi_{1,k}$. It follows that $\chi(\mathcal{B})$ is unbounded. $\square$

Theorem 3.16. *An equivalence structure $\mathcal{A} = (\omega, E)$ is generically Σ_2 if and only if it has a c.e. copy.*

Proof. Suppose that $\mathcal{A} = (\omega, E)$ is generically Σ_2 and let $\mathcal{D}$ be a dense c.e. set such that $\mathcal{D} = (D, E)$ is a c.e. structure and also a Σ_2-elementary substructure of $\mathcal{A}$. Then $\chi(\mathcal{D}$ is a Σ^0_2 set since $\mathcal{D}$ is c.e. and $\chi(\mathcal{D}) = \chi(\mathcal{A})$ since $\mathcal{D}$ is a Σ_2-elementary submodel of $\mathcal{A}$. If $\mathcal{D}$ has no infinite classes, then $char(\mathcal{D})$ has an s_1 function. Thus $\mathcal{A}$ has a computable copy whether or not it has infinite classes. If $\mathcal{D}$ has an infinite class, then $\mathcal{A}$ also has an infinite class and therefore has a c.e. copy. The other direction is immediate. $\square$

4. Coarsely computable and Coarsely Σ_1 structures

The results on generically Σ_n structures lay down a baseline for the deeper results on coarsely computable injection structures. We will show in particular that not every coarsely Σ_1 injection structure has a generically computable copy and that there are injection structures which do not have coarsely computable copies.

In this section, we define the notions of coarsely computable and coarsely Σ_1 structures. We investigate these notions for equivalence structures and for injection structures.

Definition 4.1. For any structure $\mathcal{A}$:

(1) $\mathcal{A}$ is *coarsely computable* if there is a computable structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with universe D is a substructure of both $\mathcal{A}$ and of $\mathcal{E}$ and all relations and functions agree on D.

(2) $\mathcal{A}$ is *coarsely c.e.* if there is a c.e. structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with universe D is a substructure of both $\mathcal{A}$ and of $\mathcal{E}$ and all relations and functions agree on D.
(3) $\mathcal{A}$ is *coarsely* Σ_n if there is a c.e. structure $\mathcal{E}$ and a dense set D such that the substructure $\mathcal{D}$ with universe D is a Σ_n-elementary substructure of both $\mathcal{A}$ and of $\mathcal{E}$ and all relations and functions agree on D.

We note that for $n = 1$, xxxxx

As above for the variants of generically computable structures, for $n > 0$, any any coarsely Σ_{n+1} structure is coarsely Σ_n, and any coarsely Σ_1 structure without relations is coarsely computable. Recall from Theorem 0.2 that the notions of generically computable and coarsely computable sets are incomparable. This implies that the same is true for structures.

Proposition 4.2. *There is a generically computable structure which is not coarsely computable and there is a coarsely computable structure which is not generically computable.*

Proof. First let A be a set which is generically computable structure but is not coarsely computable. Then by Example 0.3, the structure (ω, A) is generically computable structure but not coarsely computable. A similar argument works when the set A is coarsely computable but not generically computable. $\square$

Certainly every coarsely computable structure is also coarsely c.e. It is easy to see that the structure (ω, A) will be coarsely c.e. if and only if there is a c.e. set E and a dense set D such that $A \cap D = E \cap D$.

We want to compare and constrast coarsely computable (c.e., Σ_n) structures with generically computable and Σ_n structures, beginning with the following.

Proposition 4.3. *Any generically computable injection structure has a coarsely computable copy.*

Proof. Let $\mathcal{A} = (\omega, f)$ be a generically computabl injection structure. By the proof of Proposition 2.6 above, we may assume that $\mathcal{A}$ has a dense computable substructure $\mathcal{D} = (D, f)$. Then we may extend $\mathcal{D}$ to a computable structure $\mathcal{C} = (\omega, g)$ by defining $g(x) = f(x)$ for $x \in D$ and $g(x) = x$ for $x \notin D$. Then $\mathcal{D}$ is a dense computable substructure of both $\mathcal{A}$ and $\mathcal{C}$, so that $\mathcal{A}$ is coarsely computable. $\square$

It is natural to ask whether any generically computable structure actually is coarsely computable. The next result answers this question in the negative.

Theorem 4.4. *There is a generically computable injection structure which is not coarsely computable.*

Proof. Let D be an asymptotically dense simple c.e. set. This is easily constructed by adding elements to a simple c.e. set, as follows. Recall that the usual construction produces a c.e. set A which contains at most n elements which are $< 2^n$ for each n, with a single element $i > 2^t$ entering A for each e, when it enters the e'th c.e. set W_e. Just take an arbitrary dense computable set B which contains exactly $2^n - 2n$ elements $< 2^n$ for each $n > 3$ and then $D = A \cup B$ will be the dense simple c.e. set.

Now let $D = \{a_0, a_1, \dots\}$ where there is a computable one-to-one function φ with $varphi(a_i) = a_{i+1}$ for each i and define the function f on D so that $f(a_i) = a_{i+1}$.

Then f is a partial computable function which is total on the set D. That is, given $a \in D$, simply enumerate D until you see that $a = a_i$ and then output a_{i+1}. Now let K be an immune set and extend D to a generically computable structure (ω, f) by defining an injection on $\omega \setminus D$ which has exactly one orbit of size k for each $k \in K$.

We claim that $\mathcal{A} = (\omega, f)$ cannot be coarsely computable.

First we check that $\mathcal{A}$ has no other c.e. dense substructures, modulo finite. Let E be a dense c.e. set such that (E, f) is a c.e. substructure of $\mathcal{A}$. First consider $D \cap E$. This has to be cofinal in the orbit given by D and hence is either empty or is equal to D, modulo finite. Next consider $E \setminus D$. If this were infinite, then it either contains an infinite orbit, which will be a c.e. set in the complement of D, or it has infinitely many finite orbits. In that case, $\{k : (\exists x \in E)\mathcal{O}(x) = k\}$ is an infinite c.e. subset of K, contradicting the definition of f above.

Next suppose that $\mathcal{C} = (\omega, g)$ is a computable extension of $\mathcal{D}$. Since D is cofinite, the set $\omega \setminus D$ is infinite. The argument proceeds as in the last paragraph, with one extra case, that is, $\mathcal{C}$ might extend the orbit $\mathcal{O}(a_0)$ from $\mathcal{D}$ to an orbit of type $\mathbb{Z}$. But then $\{x : (\exists n > 0)g^{(n)}(x) = a_0\}$ would be an infinite c.e. set in the complement of D. Otherwise, $(\omega \setminus D, g)$ either contains an infinite oribt, or has an infinite c.e. character, either of which produces an infinite c.e. subset disjoint from D. □

The situation is somewhat different for equivalence structures. Of course we know that every equivalence structure has a generically computable copy.

Proposition 4.5. *Any generically computable equivalence structure is coarsely c.e.*

Proof. Let $\mathcal{A} = (\omega, E)$ be an equivalence structure and let D be a dense c.e. set such that $\mathcal{D} = (D, E)$ is a computable substructure of $\mathcal{A}$ Then we may extend E to a c.e. equivalence relation R on ω by letting xRy if and only if $x = y$ or $x, y \in D$ and xEy. Thus for $x \in D$, $[x]_R = [x]_E$ and for $x \notin D$, $[x]_R = \{x\}$. □

Let $\mathcal{E} = (\omega, E)$ be the canonical equivalence structure with one class of every finite size k. The equivalence classes of (ω, E) are $\{\{0\}, \{1, 2\}, \{3, 4, 5\}, \dots\}$. The first k classes have $1 + 2 + \cdots + k = k(k + 1)/2$ elements. Let K be any set and let A_K be the classes of size k for $k \in K$, under E.

Similarly let $\mathcal{C} = (\omega, f)$ be the structure with orbits $\{\{0\}, \{1, 2\}, \{3, 4, 5\}, \dots\}$, so that $f(0) = 0$, $f(1) = 2$ and $f(2) = 1$, and so on. The first k classes have $1 + 2 + \cdots + k = k(k + 1)/2$ elements.

Lemma 4.6. *If K is a dense set, then A_K is also a dense set.*

Proof. Suppose that the complement of K contains m out of the first n positive numbers. Then the classes of size k with $k \in K \cap \{1, 2, \dots, n\}$ contain at most $n + (n - 1) + \cdots + (n - m + 1) = m(2n - m + 1)/2$ elements out of a total of $1 + 2 + \cdots + n = n(n + 1)/2$. Then the ratio is $\frac{m}{n} \cdot \frac{2n-m+1}{n+1} \leq 2m/n$. Thus, if $\omega \setminus K$ has density zero, then A_K will have density 1. □

Theorem 4.7. *For any dense co-infinite set K, there is a coarsely Σ_1 equivalence structure $\mathcal{A}$ with character $\{(k, i) : k \in K, i \leq 2\}$ and no infinite classes.*

Proof. Let $\mathcal{E} = (\omega, E)$ be the canonical computable structure described above with one class of every finite size k. Let A_K be the dense subset of ω which will have character $\{k, 1) : k \in K\}$ under E. Then take $\omega \setminus A_K$ and partition it into exactly one class of size k for $k \in K$ to create the structure $\mathcal{A}$. Then $\mathcal{A}$ agrees with $\mathcal{E}$ on the dense subset A_K. (A_K, E) is a Σ_1-elementary substructure of both $\mathcal{E}$ and $\mathcal{A}$ since $\chi(A_K) = \{(k,1) : k \in K\}$ is unbounded. Thus $\mathcal{A}$ is coarsely c.e.

To obtain the coarsely computable injection structure, define an injection g which agrees with the canonical function f on the set A_K and extend this function on $\omega \setminus A_K$ to add one additional orbit of each size k for $k \in K$. Again this structure agrees with the computable structure $\mathcal{C}$ on the dense set A_K. □

Theorem 4.8. *For any dense co-infinite set K, there is a coarsely computable injection structure with character $\{(k,i) : k \in K, i \leq 2\}$ and no infinite orbits.*

Proof. The proof is similar to Theorem 4.7, except that (A_K, f) will not be a Σ_1-elementary substructure of $\mathcal{A}$ since the character is different from $\chi(\mathcal{A})$, as it has only one class of size k for $k \in K$ whereas $\mathcal{A}$ has two. □

Lemma 4.9. *There is a Π^0_1 dense set with no infinite c.e. subset and a Π^0_2 dense set K with no infinite Σ^0_2 subset.*

Proof. The notion of an *immune* set, that is a Π^0_1 set with no infinite c.e. subset, is well-studied and easily generalized. The standard proof may be modified as follows to obtain a dense set. Let $S_1, S_2, \ldots$ enumerate the Σ^0_2 sets and define K to omit the least member of S_i which is greater than 2^i. Then K must contain at least $2^i - i$ of the first 2^i numbers and hence has density one. □

Proposition 4.10. (1) *There is a coarsely computable injection structure with no generically computable copy.*
(2) *There is a coarsely Σ_1 equivalence structure with no generically computable copy.*

Proof. Let K be a dense immune set and let $\mathcal{A}$ be the injection structure with character $\{(k,i); k \in K, i \leq 2\}$ from Lemma 4.9 If $\mathcal{B}$ were a generically computable copy of $\mathcal{A}$, then $\mathcal{B}$ has no infinite classes and thus $\chi(\mathcal{B}) = \chi(\mathcal{A})$ must have an infinite c.e. subset C by Proposition 2.6. Then $\{k : (k,1) \in C \ \vee \ (k,1) \in C\}$ is an infinite c.e. subset of K, which is a contradiction. The proof for equivalence structures similarly follows from Lemma 4.9 and Theorem 3.14 □

Next we will show that there are equivalence structures which do not have coarsely c.e. copies and injection structures which have no coarsely computable copies.

Theorem 4.11. *There is an infinite Δ^0_4 set $K \subset \omega$ such that if $\mathcal{C} = (\omega, R)$ is a computably enumerable equivalence structure such that $\{x : |[x]_R| = k\}$ has asymptotic density zero for any k, and such that if D is a set of asymptotic density one, then D is not a subset of $\{x : |[x]_R| \in K\}$. Thus any equivalence structure $\mathcal{A}$ with character $\chi(\mathcal{A}) \subset K \times \{1\}$ cannot be coarsely c.e.*

Proof. Let $\mathcal{C}_e := (\omega, S_e)$ be the e^{th} computably enumerable equivalence structure. That is, let W_e be the e^{th} computably enumerable set, and let S_e be the reflexive, symmetric, transitive closure of $\{(x,y) : \langle x,y\rangle \in W_e$. Let $[x]_e$ denote the equivalence class of x in $\mathcal{C}_e$. We need to meet the following requirements.

Requirement R_e: If $\{x : |[x]_e| = k\}$ has asymptotic density zero for all k, then $\{x : |[x]_e| \in K\}$ does not have asymptotic density one.

We begin the construction with $K^0 = \omega$ and remove numbers at certain stages to accomplish the requirements. At the same time, we need to ensure that K

is infinite. So the construction will preserve an element of K each time that it removes an infinite number of elements. We may assume for the construction that $\{x : [x]_e$ is infinite$\}$ has upper density zero, otherwise the conclusion is immediate.

We will show how to satisfy an individual requirement by the case $e = 0$. Let $\mathcal{C} = (\omega, S_0)$, let $S = S_0$, and consider the four sets $A_i = \{x : |[x]_S| = i \ mod\ 4\}$ for $i = 0, 1, 2, 3$. Since the union of the sets equals ω, at least one of the sets, say A_j, must have upper asymptotic density at least $1/4$. Let us suppose that $\{x : |[x]_S| \ = k\}$ has asymptotic density zero for all k, so that we need to take action on requirement R_0. Then we will ensure that $K \cap \{i : i = j \ mod\ 4\} = \{4+j\}$; that is, we let $K^1 = \{4+j\} \cup \{k : k \neq j \ mod\ 4\}$ and maintain $K \cap \{i : i = j \ mod\ 4\} = \{4+j\}$ throughout the construction. Then $\{x : |[x]_S| \in K\}$ must have density at most $3/4$, so that it cannot contain any set D has asymptotic density one.

The general construction of K is in stages. After stage e, we will have designated, for certain $i \leq e$, a value $j(i)$ and corresponding set $A_i = \{x : |[x]_i| = j(i) \ mod\ 2^{i+2}\}$, so that for $i \neq h$, we have $A_i \cap A_h = \emptyset$. We will have removed $K_i = \{m : m = j(i) \ mod\ 2^{i+2}\}$ from K, except for $2^{i+2} + j(i)$, for such i, resulting in the set K^s. Note that we will have removed at most one set K_i mod 2^{i+2} for each $i \leq e$, for a total of at most $2^e + 2^{e-1} + \cdots + 1 < 2^{e+1}$ classes mod 2^{e+2}, resulting in the set K^e. Thus, there remain 2^{e+1} classes mod 2^{e+2} to work with, each disjoint from the previous classes. At stage $e+1$, we will ensure Requirement R_e (if necessary) by removing a set of class sizes from K. If there exists k such that $\{x : |[x]_{e+1}| = k\}$ has positive measure, then we take no action. If not, then we select $j = j(e+1) < 2^{e+3}$ such that $A_{e+1} = \{x : |[x]|_{e+1} = j \ mod\ 2^{e+3}\}$ has upper density at least 2^{-e-3} and we let $K_{e+1} = \{m : m = j(e+1) \ mod\ 2^{e+3}\}$. If K_{e+1} meets one of the previous classes K_i, then in fact $K_{e+1} \subset K_i$, so that we have already removed all but one element of K_{e+1} from K by stage s. Otherwise, we remove $K_{e+1} = \{m : m = j \ mod\ 2^{e+3}\}$ from K^e, except for $2^{e+3} + j$, to obtain K^{e+1}.

Let $K = \cap_s K^s$. It remains to check that K satisfies each Requirement R_e and is an infinite set.

First we show that action is taken infinitely often. Suppose, by way of contradiction, that no action is taken after stage e. Then K will consist of a finite number of equivalence classes modulo 2^{e+2} plus a finite set. Thus K will be computable. Hence there is some i such that $\mathcal{C}_i$ consists of exactly one class of size k for each $k \in K$. Thus at stage i, when we select j such that $\{x : |[x]_i| = j \ mod\ 2^{i+2}\}$ has positive upper density in $\mathcal{C}_i$, and consider $K_i = \{m : m = j \ mod\ 2^{i+2}\}$, we would have $K_i \subset K \subset K^{i+1}$. But then we would have taken action and removed all but one value of K_i from K.

Next we need to check that K is infinite. Since action was taken infinitely often, we have preserved in K an element $2^{i+2} + j(i)$ of K_i for infinitely many i. Since the sets $\{K_i : i \in \omega\}$ are disjoint, this element is never removed at any later stage. Hence K is infinite.

Now suppose that $\{x : |[x]_e| \ = k\}$ has asymptotic density zero for all k, and suppose, by way of contradiction, that $\{x : |[x]_e| \in K\}$ has asyptotic density one. Then at stage e of the construction we will have selected $j < 2^{e+2}$ such that $A_j = \{x : |[x]|_e = j \ mod\ 2^{e+2}\}$ has upper density at least 2^{-e-2}, and defined

$$K_e = \{m : m = j \ mod\ 2^{e+2}\}$$

. Since $K \subset K^{e-1}$, it follows that K_e is disjoint from all previous K_i. So we will remove all but one element of K_e from K at stage e. It follows that $\{x : |[x]_e| \in K\}$ has lower density at most $1 - 2^{-e-2}$.

Finally, suppose that $\mathcal{A} = (\omega, S)$ has character $\chi(\mathcal{A}) \subseteq K \times \{1\}$ and is coarsely c.e. Let $\mathcal{C} = (\omega, R)$ be a computable equivalence structure,say $R = S_e$. Let D be an S-faithful, R-faithful set of density one such that R and S agree on D. Since D is S-faithful, $D \subseteq \{x : |[x]|_S \in K\}$. Since R and S agree on D, and D is R-faithful, it follows that $D \subseteq \{x : |[x]|_e \in K\}$. By the assumption on $\mathcal{C}$, this means that $\{x : [x]_e = k\}$ has density zero for each k. It follows from Requirement R_e that $\{x : |[x]_e| \in K\}$ does not have asymptotic density one. But this contradicts the fact that the subset D has density one.

An upper bound on the complexity of K may be determined as follows. First, we observe that $\{x : |[x]_i = j\}$ is uniformly Σ^0_2 and thus $C(i,j,e) = \{x : |[x]_i = j \ mod \ 2^e\}$ is also uniformly Σ^0_2. Then the lower density $\delta(C(i,j,e) \geq \frac{1}{4}$ if and only if

$$(\forall m)(\exists n > geqm)|C(i,j,e) \cap n| \geq \frac{n}{4}.$$

Thus this test is Π^0_3. So the construction may be done using an oracle for O'''. So the set K_i is uniformly computable in O'''. Since K is the intersection of the sequence $(K_i)_i$, it follows that K is a Π^0_4 set. $\square$

Here is the injection structure result.

Theorem 4.12. *There is an infinite set $K \subset \omega$ such that if $\mathcal{C} = (\omega, f)$ is a computable injection structure such that $\{x : |\mathcal{O}_f(x)| \ = k\}$ has asymptotic density zero for any k, and if D is a set of asymptotic density one, then D is not a subset of $\{x : |\mathcal{O}_f(x)| \in K\}$. Thus any injection structure $\mathcal{A}$ with character $\chi(\mathcal{A}) \subset K \times \{1\}$ cannot be coarsely computable.*

Proof. Here we let $\mathcal{C}_e := (\omega, S_e)$ be the e^{th} *potential* computabe injection structure. That is, let W_e be the e^{th} computably enumerable set, and let $f_e(x)$ be the least y such that $\langle x, y\rangle \in W_e$, if any. Let $\mathcal{O}_e(x)$ be the orbit of x under f, if defined. Then we need to meet the following requirements.

Requirement R_e: If $\mathcal{C}_e$ is an injection structure and $\{x : |\mathcal{O}_e(x)| \ = k\}$ has asymptotic density zero for all k, then $\{x : |\mathcal{O}_e(x)| \in K\}$ does not have asymptotic density one.

We begin the construction with $K^0 = \omega$ and remove numbers at certain stages to accomplish the requirements. At the same time, we need to ensure that K is infinite. So the construction will preserve an element of K each time that it removes an infinite number of elements. We may assume for the construction that $\{x : \mathcal{O}_e(x) \text{ is infinite}\}$ has upper density zero, otherwise the conclusion is immediate.

We will show how to satisfy an individual requirement by the case $e = 0$. Let $\mathcal{C} = (\omega, S_0)$, let $S = S_0$, and consider the four sets $A_i = \{x : |\mathcal{O}_0(x)| = i \ mod \ 4\}$ for $i = 0, 1, 2, 3$. Since the union of the sets equals ω, at least one of the sets, say A_j, must have upper asymptotic density at least $1/4$. Let us suppose that $\{x : |\mathcal{O}_e(x)| \ = k\}$ has asymptotic density zero for all k, so that we need to take action on requirement R_0. Then we will ensure that $K \cap \{i : i = j \ mod \ 4\} = \{4 + j\}$; that is, we let $K^1 = \{4+j\} \cup \{k : k \neq j \ mod \ 4\}$ and maintain $K \cap \{i : i = j \ mod \ 4\} = \{4+j\}$ throughout the construction. Then $\{x : |[x]_S| \in K\}$ must have density at most $3/4$, so that it cannot contain any set D which has asymptotic density one.

The details of the construction are similar to those given in the proof of Theorem 4.11 and are therefore omitted here. An upper bound on the complexity of K may be determined as follows. First, we observe that $\{x : |\mathcal{O}_i(x) = j\}$ is uniformly Σ^0_1 and thus $C(i, j, e) = \{x : |\mathcal{O}_i(x) = j \ mod \ 2^e\}$ is also uniformly Σ^0_1. Then the lower density $\delta(C(i, j, e) \geq \frac{1}{4}$ if and only if

$$(\forall m)(\exists n > geqm)|C(i, j, e) \cap n| \geq \frac{n}{4}.$$

Thus this test is Π^0_2 So the construction may be done using an oracle for O'', and it follows that the K is a Π^0_3 set. □

As was the case for generically Σ_1 structures, any coarsely Σ_1 structure is always isomorphic to a computable structure.

Proposition 4.13. *The following are equivalent for any injection structure $\mathcal{A} = (\omega, f)$:*

(a) *$\mathcal{A}$ has a coarsely Σ_1 copy;*
(b) *$\chi(\mathcal{A})$ is a c.e. set;*
(c) *$\mathcal{A}$ has a computable copy.*

Proof. Suppose first that $\mathcal{A} = (\omega, f)$ is a coarsely Σ_1 injection structure. Let $\mathcal{B} = (\omega, g)$ be a c.e. structure and D be a dense set such that $f = g$ on the set D and such that $\mathcal{D} = (D, f)$ is a Σ_1-elementary substructure of both sA and $\mathcal{B}$. Then $\chi(\mathcal{A}) = \chi(\mathcal{D}) = \chi(\mathcal{B}$ and is therefore a c.e. set. The next implication follows from Proposition 2.4. □

Here is the similar result for equivalence structures.

Proposition 4.14. *The following are equivalent for any equivalence structure $\mathcal{A} = (\omega, E)$:*

(a) *$\mathcal{A}$ is coarsely Σ_2;*
(c) *$\mathcal{A}$ has a c.e. copy.*

We have only a partial result for coarsely Σ_2 structures.

Proposition 4.15. *Let $\mathcal{A}$ be an equivalence structure with an infinite class. Then $\mathcal{A}$ is coarsely Σ_2 if and only if $\mathcal{A}$ has a c.e. copy.*

Proof. Suppose first that $\mathcal{A} = (\omega, E)$ is a coarsely Σ_2 equivalence structure. Let $\mathcal{B} = (\omega, R)$ be a c.e. structure and D be a dense set such that $E = R$ on the set D and such that $\mathcal{D} = (D, E)$ is a Σ_2-elementary substructure of both sA and $\mathcal{B}$. Then by Proposition 3.15 $\chi(\mathcal{A}) = \chi(\mathcal{D}) = \chi(\mathcal{B}$ and is therefore a Σ^0_2 set. Since $\mathcal{A}$ has an infinite class, it follows from Proposition 3.4 that $\mathcal{A}$ is isomorphic to a c.e. structure by Proposition 3.4. The other implication is immediate. □

5. Conclusion and Future Research

In this paper, we have introduced some notions of generically computable and coarsely computable structures. For injection structures and equivalence structures, we have characterized the generically computable, generically Σ_1 and generically Σ_1 structures. Next we will show that there are equivalence structures which do not have coarsely c.e. copies and injection structures which have no coarsely computable copies.

We are continuing to work on these notions for Abelian p-groups, following up on Example 0.7. We are also exploring the notions of generically and coarsely computable isomorphisms. So far we have shown that there are computable structures which are not computably isomorphic but which have a coarsely computable isomorphism.

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