Index Sets of Computable Structures

Wesley Calvert*
Department of Mathematics
University of Notre Dame
wcalvert@nd.edu

Valentina S. Harizanov
Department of Mathematics
George Washington University
harizanv@gwu.edu

Julia F. Knight
Department of Mathematics
University of Notre Dame
knight.1@nd.edu

Sara Miller
Department of Mathematics
University of Notre Dame
smiller9@nd.edu

May 27, 2005

Abstract

The index set of a computable structure $\mathcal{A}$ is the set of all indices for computable isomorphic copies of $\mathcal{A}$. We determine, using the arithmetical hierarchy and the difference hierarchy, the exact complexity of the index sets of structures within the following classes of structures: finite structures, vector spaces, Archimedean ordered fields, Abelian p-groups, and the models of the original Ehrenfeucht theory.

1 Introduction

One of the goals of computable structure theory is to study the relationship between algebraic and algorithmic properties of structures. Our languages are computable, and our structures have universe contained in ω, which we think of as a computable set of constants. In measuring complexity, we identify a structure $\mathcal{A}$ with its atomic diagram, $D(\mathcal{A})$, *via* Gödel coding. In particular, $\mathcal{A}$ is *computable* if $D(\mathcal{A})$ is computable. For a computable structure $\mathcal{A}$, an *index* is a number a such that $\varphi_a = \chi_{D(\mathcal{A})}$, where $(\varphi_a)_{a\in\omega}$ is a computable enumeration of all unary partial computable functions. The *index set* for $\mathcal{A}$ is the set $I(\mathcal{A})$ of all indices for computable (isomorphic) copies of $\mathcal{A}$. For a class K of structures, closed under isomorphism, the *index set* is the set $I(K)$ of all indices for computable members of K. There is quite a lot of work on index sets [14], [6], [2], [3], [5], [8], [20], [21], [7], etc. In this paper, we present evidence for the following thesis:

*The first author was partially supported by NSF grants DMS 0139626 and DMS 0353748

For a given computable structure $\mathcal{A}$, to calculate the precise complexity of $I(\mathcal{A})$, we need a good description of $\mathcal{A}$, and once we have an "optimal" description, we the complexity of $I(\mathcal{A})$ will match that of the description.

Our evidence for this thesis consists of calculations for computable structures of several familiar kinds: finite structures, $\mathbb{Q}$-vector spaces, Archimedean ordered fields, reduced Abelian p-groups of length less than ω^2, and models of the original Ehrenfeucht theory.

We should say what qualifies as a "description" of a structure, and how we measure the complexity. The Scott Isomorphism Theorem says that for any countable structure $\mathcal{A}$ there is a sentence of $L_{\omega_1\omega}$ whose countable models are exactly the isomorphic copies of $\mathcal{A}$ (see [11]). Such a sentence is called a *Scott sentence* for $\mathcal{A}$. A Scott sentence for $\mathcal{A}$ certainly describes $\mathcal{A}$.

There is earlier work [16], [15] investigating subsets of the Polish space of structures with universe ω for a given countable language. Concerning the possible complexity (in the non-effective Borel hierarchy) of the set of copies of a given structure, it is shown in [16] that if the set is $\mathbf{\Delta^0_{\alpha+1}}$, then it is d-$\mathbf{\Sigma^0_\alpha}$. In [15] it is shown that the set cannot be properly $\mathbf{\Sigma^0_2}$. There are examples illustrating other possibilities.

Most of the structures we consider follow one of two patterns. Either there is a computable Π_n Scott sentence, and the index set is m-complete Π^0_n, or else there is a Scott sentence which is computable "d-Σ_n" (the conjunction of a computable Σ_n sentence and computable Π_n sentence), and the index set is m-complete d-Σ_n. For example, a computable reduced Abelian p-group of length ω has a computable Π_3 Scott sentence, and the index set is m-complete Π^0_3. A $\mathbb{Q}$-vector space of finite dimension at least 2 has a Scott sentence that is computable d-Σ_2, and the index set is m-complete d-Σ^0_2. The "middle model" of the original Ehrenfeucht theory illustrates a further pattern. There is a computable Σ_3 Scott sentence, and the index set is m-complete Σ^0_3. Often the first Scott sentence that comes to mind is not optimal. In some cases, in particular, for some of the Abelian p-groups, it requires effort to show that a certain sentence of a simpler form actually is a Scott sentence.

For some structures, we obtain more meaningful results by locating the given computable structure $\mathcal{A}$ within some natural class K. We say how to describe $\mathcal{A}$ within K, and also how to calculate the complexity of $I(\mathcal{A})$ within K.

Definition 1.1. A sentence φ is a Scott sentence for $\mathcal{A}$ *within* K if the countable models of φ in K are exactly the isomorphic copies of $\mathcal{A}$.

The following definitions were used already in [3].

Definition 1.2. Let Γ be a complexity class (e.g., Π^0_3).

1. $I(\mathcal{A})$ is Γ *within* K if $I(\mathcal{A}) = R \cap I(K)$ for some $R \in \Gamma$.

2. $I(\mathcal{A})$ is *m-complete* Γ *within* K if $I(\mathcal{A})$ is Γ within K and for any $S \in \Gamma$, there is a computable function $f : \omega \to I(K)$ such that

$$n \in S \text{ iff } n \in I(\mathcal{A}) \ .$$

that is, there is a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ such that

$$n \in S \text{ iff } \mathcal{C}_n \cong \mathcal{A}.$$

Example 1. Let $\mathcal{A}$ be the field with 3 elements, and let K be the class of finite prime fields. There is a Scott sentence for $\mathcal{A}$ within K saying $1+1+1=0$. The index set for $\mathcal{A}$ is computable *within* K.

The example above is an exception. In most of the examples we consider, even when we locate our structure within a class K, the optimal description is a true Scott sentence, but the context helps us calculate the complexity of the index set in a meaningful way.

Example 2. Let $\mathcal{A}$ be a linear ordering of size 3, and let K be the class of linear orderings. There is a computable d-Σ_1 Scott sentence saying that there are at least 3 elements ordered by the relation, and not more. We will show that the index set for $\mathcal{A}$ is m-complete d-c.e. *within* K.

Here we mention some related work. The proof of the Scott Isomorphism Theorem leads to an assignment of ordinals to countable structures. By a result of Nadel [17], for any hyperarithmetical structure, there is a computable infinitary Scott sentence iff the Scott rank is computable. There are several different definitions of Scott rank in use. Since we are more interested in Scott sentences, we shall not give any of them.

Work on index sets for particular computable structures is related to work on isomorphism problems for classes of computable structures [2], [3], [8]. The *isomorphism problem* for a class K is the set $E(K)$ consisting of pairs (a, b) of indices for computable members of K that are isomorphic. It is often the case that for the classes K for which the complexity of the isomorphism problem is known, there is a single computable $\mathcal{A} \in K$ such that the index set for $\mathcal{A}$ has the same complexity as $E(K)$. Results on index sets are useful in other contexts as well. In [4], they are used in connection with Δ^0_2 categoricity of computable structures.

We consider finite structures in Section 2, vector spaces in Section 3, Archimedean ordered fields in Section 3, Abelian p-groups in Section 4, and models of the original Ehrenfeucht theory in Section 5.

2 Finite Structures

Finite structures are the easiest to describe. It is perhaps surprising that there should be any variation in complexity of index sets for different finite structures, and, indeed, there is almost none. In the following theorem, we break with convention by allowing a structure to be empty.

Theorem 2.1. *Let L be a finite relational language. Let K be the class of finite L-structures, and let $\mathcal{A} \in K$.*

1. *If $\mathcal{A}$ is empty, then $I(\mathcal{A})$ is m-complete Π^0_1 within K.*

2. *If $\mathcal{A}$ has size $n \geq 1$, then $I(\mathcal{A})$ is m-complete d-c.e. within K.*

Proof. For 1, first note that $\mathcal{A}$ has a finitary Π_1 Scott sentence saying that there is no element. From this, it is clear that $I(\mathcal{A})$ is Π^0_1 within K. For completeness, let $\mathcal{B}$ be an L-structure with just one element. For an arbitrary Π^0_1 set S, we can produce a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that

$$\mathcal{A}_n \cong \begin{cases} \mathcal{A} & \text{if } n \in S \ , \\ \mathcal{B} & \text{if } n \notin S \ . \end{cases}$$

For 2, we have a finitary existential sentence φ stating that there is a substructure isomorphic to $\mathcal{A}$, and another finitary existential sentence ψ stating that there are at least $n+1$ elements. Then φ & $\neg\psi$ is a Scott sentence for $\mathcal{A}$. It follows that $I(\mathcal{A})$ is d-c.e. within K. For completeness, let $S = S_1 - S_2$, where S_1 and S_2 are c.e. We have the usual finite approximations $S_{1,s}$, $S_{2,s}$.

Let $\mathcal{A}^-$ be a proper substructure of $\mathcal{A}$, and let $\mathcal{A}^+$ be a finite proper superstructure of $\mathcal{A}$. We will build a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that

$$\mathcal{A}_n \cong \begin{cases} \mathcal{A}^- & \text{if } \ n \notin S_1 \ , \\ \mathcal{A} & \text{if } n \in S_1 - S_2 \ , \\ \mathcal{A}^+ & \text{if } n \notin S_1 \cap S_2 \ . \end{cases}$$

To accomplish this, let $D_0 = D(\mathcal{A}^-)$. At stage s, if $n \notin S_{1,s}$, we let D_s be the atomic diagram of $\mathcal{A}^-$. If $n \in S_{1,s} - S_{2,s}$, we let D_s be the atomic diagram of $\mathcal{A}$. If $n \in S_{1,s} \cap S_{2,s}$, we let D_s be the atomic diagram of $\mathcal{A}^+$. There is some s_0 such that for all $s \geq s_0$, $n \in S_1$ iff $n \in S_{1,s}$, and $n \in S_2$ iff $n \in S_{2,s}$. Let $\mathcal{A}_n$ be the structure with diagram D_s for $s \geq s_0$. It is clear that $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$. □

In the introduction, we mentioned finite prime fields. Let p be a prime number. By the results above, $\mathbb{F}_p$ has a Scott sentence of form φ & $\neg\psi$, where φ and ψ are finitary Σ_1. It follows that $I(\mathbb{F}_p)$ has a Scott sentence which is the conjunction of a finitary existential sentence and a finitary universal sentence. Then the index set is d-c.e. within the class of finite structures for the appropriate language. However, within the class of finite prime fields, $\mathbb{F}_p$ has a finitary quantifier-free Scott sentence saying $p \cdot 1 = 0$. Then $I(\mathbb{F}_p)$ is computable within that class.

3 Vector Spaces

The finite dimensional vector spaces over a fixed field are completely determined by a finite set (a basis), so we might expect these to behave much like finite structures. However, we have added complexity because of the fact that for $1 \leq m < n$, if $\mathcal{V}_n$ is a space of dimension n, and $\mathcal{V}_m$ is an m-dimensional subspace, if $\mathcal{V}_n \models \varphi(\bar{c}, \bar{a})$, where φ is finitary quantifier-free, and $\bar{c}$ is in $\mathcal{V}_m$, then there exists $\overline{a}'$ such that $\mathcal{V}_m \models \varphi(\overline{c}, \overline{a}')$. We work with vector spaces over $\mathbb{Q}$, for concreteness, but any other infinite computable field would give exactly the same results.

Proposition 3.1. *Let K be the class of vector spaces over $\mathbb{Q}$, and let $\mathcal{A}$ be a member of K.*

1. *If $\dim(\mathcal{A}) = 0$, then $I(\mathcal{A})$ is m-complete Π^0_1 within K.*
2. *If $\dim(\mathcal{A}) = 1$, then $I(\mathcal{A})$ is m-complete Π^0_2 within K.*
3. *If $\dim(\mathcal{A}) > 1$, then $I(\mathcal{A})$ is m-complete d-Σ^0_2 within K.*

Proof. For 1, first we note that $\mathcal{A}$ has a a finitary Π_1 Scott sentence, within K, saying $(\forall x)\, x = 0$. It follows that $I(\mathcal{A})$ is Π^0_1 within K. Toward completeness, let S be a Π^0_1 set. We build a uniformly computable sequence of structures $(\mathcal{A}_n)_{n\in\omega}$ such that

$$\dim(\mathcal{A}_n) = \begin{cases} 0 & \text{if } n \in S\ , \\ 1 & \text{if } n \notin S\ . \end{cases}$$

Let $\mathcal{V}_0$ be a space of dimension 0, and let $\mathcal{V}_1$ be a computable extension having dimension 1. We have a computable sequence $(S_s)_{s\in\omega}$ of approximations for S such that $n \in S$ iff for all s, $n \in S_s$, and if $n \notin S_s$, then for all $t > s$, $n \notin S_t$. If $n \in S_s$, we let $D_s = D(\mathcal{V}_0)$. If $n \notin S_s$, then we let D_s consist of the first s sentences of $D(\mathcal{V}_1)$. This completes the proof for 1.

Next, we turn to 2. First, we show that $\mathcal{A}$ has a computable Π_2 Scott sentence. We have a computable Π_2 sentence characterizing the class K. We take the conjunction of this with the sentence saying

$$(\exists x)\ x \neq 0\ \&\ (\forall x)\,(\forall y)\ \bigvee\!\!\!\bigvee_{\lambda\in\Lambda} \lambda(x, y) = 0\ ,$$

where Λ is the set of all non-trivial linear combinations $q_1 x + q_2 y$, for $q_i \in \mathbb{Q}$. Now, $I(\mathcal{A})$ is Π^0_2. We do not need to locate $\mathcal{A}$ within K, since the set of indices for members of K is Π^0_2.

For completeness, let S be a Π^0_2 set. We build a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that

$$\dim(\mathcal{A}_n) = \begin{cases} 1 & \text{if } n \in S\ , \\ 2 & \text{if } n \notin S\ . \end{cases}$$

We have computable approximations S_s for S such that $n \in S$ iff for infinitely many s, $n \in S_s$.

Let $\mathcal{V}^+$ be a two dimensional computable vector space and let $\mathcal{V}$ be a 1-dimensional subspace. Let B be an infinite computable set of constants, for the universe of all $\mathcal{A}_n$. At each stage s, we have a finite partial $1-1$ function p_s from B to a target structure $\mathcal{V}$ (if $n \in S_s$) or $\mathcal{V}^+$ (if $n \notin S_s$), and we have enumerated a finite set D_s (a part of the atomic diagram of $\mathcal{A}_n$) such that p_s

maps the constants mentioned in D_s into the target structure so as to make all of the sentences true. We arrange that if $n \in S$, then $\cup_{n \in S_s} p_s$ maps B onto $\mathcal{V}$. If $n \notin S$, then there is some s_0 such that for all $s \geq s_0$, $n \notin S_s$. In this case, $\cup_{s \geq s_0} p_s$ for $s \geq s_0$ maps B onto $\mathcal{V}^+$. For all s (whether or not $n \in S_s$), D_s decides the first s atomic sentences involving constants in $dom(p_s)$.

We start with $p_0 = \emptyset$, and $D_0 = \emptyset$. Without loss of generality, we may suppose that $n \in S_0$. We consider p_0 to be mapping into $\mathcal{V}$. If there is no change in our guess about whether $n \in S$ at stage $s+1$, then $p_{s+1} \supseteq p_s$, where the first $s+1$ constants from B are in the domain, and the first $s+1$ elements of the target structure are in the range. We must say what happens when we change our guess at whether $n \in S$.

There are two cases. First, suppose $n \in S_{s+1}$ and $n \notin S_s$. In this case, we take the greatest stage $t \leq s$ such that $n \in S_t$. We let $p_{s+1} \supseteq p_t$ such that p_{s+1} makes the sentences of D_s true in $\mathcal{V}$, extending so that the first $s+1$ constants from B are in the domain, and the first $s+1$ elements of $\mathcal{V}$ are in the range. Now, say $n \notin S_{s+1}$ and $n \in S_s$. In this case, we do not look back at any earlier stage. We let $p_{s+1} \supseteq p_s$, extending so that the first $s+1$ constants from B are in the domain, and the first $s+1$ elements of $\mathcal{V}^+$ are in the range. In either case, we let $D_{s+1} \supseteq D_s$ so that for the first $s+1$ atomic sentences β involving constants in the domain of p_{s+1}, D_{s+1} includes $\pm\beta$, whichever is made true by p_{s+1}. This completes the proof for 2.

Finally, we turn to 3. Suppose $\mathcal{A}$ has dimension k, where $k > 1$. Then $\mathcal{A}$ has a d-Σ_2 Scott sentence. We take the conjunction of the axioms for $\mathbb{Q}$-vector spaces, and we add a sentence saying that there are at least k independent elements, and that there are not at least $k+1$. Then $I(\mathcal{A})$ is d-Σ^0_2. Toward completeness, let $S = S_1 - S_2$, where S_1 and S_2 are both Σ^0_2. Let $\mathcal{V}^+$ be a computable vector space of dimension $k+1$, and let

$$\mathcal{V}^- \subseteq \mathcal{V} \subseteq \mathcal{V}^+,$$

where $\mathcal{V}^-$ has dimension $k-1$, and $\mathcal{V}$ has dimension k. We will produce a uniformly computable sequence of structures $(\mathcal{A}_n)_{n\in\omega}$ such that

$$\dim(\mathcal{A}_n) = \begin{cases} k-1 & \text{if } n \notin S_1 , \\ k & \text{if } n \in S_1 - S_2 , \\ k+1 & \text{if } n \in S_1 \cap S_2 . \end{cases}$$

The construction is similar to that for 2. We have computable approximations $S_{1,s}$ and $S_{2,s}$ for S_1 and S_2, such that

$$n \in S_i \text{ iff for all but finitely many } s, n \in S_{i,s} .$$

Let B be an infinite computable set of constants, for the universe of all $\mathcal{A}_n$. At each stage s, we have a finite partial $1-1$ function p_s from B to $\mathcal{V}^-$ if $n \notin S_{1,s}$, to $\mathcal{V}$ if $n \in S_{1,s} - S_{2,s}$, and to $\mathcal{V}^+$ if $n \in S_{1,s} \cap S_{2,s}$. We have D_s, a finite part of $D(\mathcal{A}_n)$ such that p_s makes D_s true in the target structure.

We arrange that if $n \notin S_1$, then the union of the p_s for $n \notin S_{1,s}$ maps B onto $\mathcal{V}^-$. If $n \in S_1 - S_2$, then after some stage s_0, $n \in S_{1,s}$ holds, while $n \in S_{2,s}$ infinitely often. In this case, the union of p_s for $s \geq s_0$ such that $n \in S_{2,s}$ maps B onto $\mathcal{V}$. If $n \in S_1 \cap S_2$, then for some stage s_1, for $s \geq s_1$, $n \in S_{1,s} \cap S_{2,s}$, and the union of p_s for $s \geq s_1$ maps B onto $\mathcal{V}^+$.

We may suppose that $n \notin S_{1,0}$, so the target structure at stage 0 is $\mathcal{V}^-$. At stage $s+1$, if there is no change in the target structure, then we extend p_s. We must say what to do when we change our mind about the target structure. First, suppose the change is because of S_1. If $n \in S_{1,s}$ and $n \notin S_{1,s+1}$, then the target structure changes from $\mathcal{V}$ or $\mathcal{V}^+$ back to $\mathcal{V}^-$. We take the greatest stage $t < s$ such that $n \notin S_{1,t}$. We let $p_{s+1} \supseteq p_t$ such that p_{s+1} makes D_s true in $\mathcal{V}^-$, extending so that the first $s+1$ constants from B are in the domain, and the first $s+1$ elements of $\mathcal{V}$ are in the range. If $n \notin S_{1,s}$ and $n \in S_{1,s+1} - S_{2,s+1}$, then the target structure changes from $\mathcal{V}^-$ to $\mathcal{V}$. We let $p_{s+1} \supseteq p_s$, extending so that the first $s+1$ constants from B are in the domain and the first $s+1$ elements of $\mathcal{V}$ are in the range.

Now, suppose the change is because of S_2. We suppose that $n \in S_{1,s}$ and $n \in S_{1,s+1}$. If $n \in S_{2,s}$ and $n \notin S_{2,s+1}$, then the target structure changes from $\mathcal{V}^+$ back to $\mathcal{V}$. We take the greatest stage $t \leq s$ such that the target structure is $\mathcal{V}$, and we have had $n \in S_{1,s'}$ for all $t < s' < s$. If there is no such t, then we take the greatest $t < s$ such that $n \notin S_{1,t}$. We let $p_{s+1} \supseteq p_t$ such that p_{s+1} makes D_s true in $\mathcal{V}$, extending to include the first $s+1$ elements of B in the domain and the first $s+1$ elements of $\mathcal{V}$ in the range. We let $D_{s+1} \supseteq D_s$ so that for for the first $s+1$ atomic sentences β involving constants in the domain of p_{s+1}, D_{s+1} includes $\pm\beta$, whichever is made true by p_{s+1}. This completes the proof of 3.

□

The final case, that of an infinite dimensional space, is already known [2]. We include it here for completeness.

Proposition 3.2. *Let K be the class of computable vector spaces over $\mathbb{Q}$, and let $\mathcal{A}$ be a member of K of infinite dimension. Then $I(\mathcal{A})$ is m-complete Π^0_3 within K.*

Proof. We have a computable Π_3 Scott sentence for $\mathcal{A}$, obtained by taking the conjunction of the axioms for $\mathbb{Q}$-vector spaces and the conjunction over all $k \in \omega$ of computable Σ_2 sentences saying that the dimension is at least k. Therefore, $I(\mathcal{A})$ is Π^0_3. For completeness, let Cof denote the set of indices for co-finite c.e. sets. It is well known that the complement of Cof is Π^0_3 complete (see [18]). We build a uniformly computable sequence of vector spaces $(\mathcal{A}_n)_{n\in\omega}$ such that $\mathcal{A}_n$ has infinite dimension iff $n \notin Cof$.

Let $\mathcal{V}$ be an infinite dimensional vector space with basis $\{v_i : i \geq -1\}$. Let B be an infinite computable set of constants, for the universe of all $\mathcal{A}_n$. For each set $S \subseteq \omega$, let $\mathcal{V}_S$ be the linear span of $\{v_{-1}\} \cup \{v_i : i \in S\}$. Our goal is to make $\mathcal{A}_n \cong \mathcal{V}_S$, where $S = \omega - W_n$. At stage s, we have a finite approximation of S, as follows. Let $S_0 = \emptyset$. If $W_{n+1,s+1}$ includes some $x \in S_s$, we let S_{s+1}

be the result of removing from S_s all $y \geq x$. If $W_{n+1,s+1}$ contains no elements of S_s, we let S_{s+1} be the result of adding to S_s the first element of ω not in $W_{n+1,s+1}$. Note for each k, there exists s such that for all $t \geq s$, $S \cap k = S_t \cap k$. Moreover, for each s, there is some $t \geq s$ such that $S_t \subseteq \omega - W_n$. For such t, for all $t' \geq t$, $S'_t \supseteq S_t$.

For the construction, at stage s we have a finite partial $1-1$ function p_s from B into $\mathcal{V}_{S_s}$. We include the first s elements of B in the domain and the first s elements of ω that are in $\mathcal{V}_{S_s}$ in the range. We also have D_s deciding the first s atomic sentences with constants in $dom(p_s)$, such that p_s makes the sentences true in $\mathcal{V}_{S_s}$. We start with $p_0 = \emptyset$, $D_0 = \emptyset$, and we think of p_0 as mapping into $\mathcal{V}_{S_0}$.

At stage $s+1$, we define p_{s+1} as follows. First, suppose S_{s+1} is the result of adding an element to S_s. Then $p_{s+1} \supseteq p_s$ including the first $s+1$ elements of B in the domain, and the first $s+1$ elements of ω that are in $\mathcal{V}_{s+1}$ in the range. Now, suppose S_{s+1} is the result of removing one or more elements from S_s, so that $S_{s+1} = S_t$ for some greatest $t < s$. We take $p_{s+1} \supseteq p_t$ such that p_{s+1} makes D_s true in $\mathcal{V}_{s+1} = \mathcal{V}_t$. We let D_{s+1} extend D_s so as to decide the first $s+1$ atomic sentences involving constants in $dom(p_{s+1})$. This completes the construction.

There is an infinite sequence of stages t such that for all $s > t$, $p_s \supseteq p_t$. Let f be the union of the functions p_t for these t. We can see that f is a $1-1$ mapping of B onto $\mathcal{V}_S$. Moreover, if $\mathcal{A}_n \cong_f \mathcal{V}_S$, then $\cup_s D_s$ is the atomic diagram of $\mathcal{A}_n$. □

We have completely characterized the m-degrees of index sets of computable vector spaces over $\mathbb{Q}$, within the class of all such vector spaces.

4 Archimedean Ordered Fields

Archimedean ordered fields are isomorphic to subfields of the reals. They are determined by the Dedekind cuts that are filled.

Theorem 4.1. *Let K be the class of Archimedean ordered fields. If $\mathcal{A}$ is either $\mathbb{Q}$, or $\mathbb{Q}(a)$ for a real algebraic number a, then $I(\mathcal{A})$ is m-complete Π^0_2 within K.*

Proof. For $\mathbb{Q}$, we have a computable Π_2 Scott sentence describing an ordered field in which each x is equal to a ratio of integers. For $\mathbb{Q}(a)$, where $p(x)$ is the minimal polynomial with root a, we have a computable Π_2 Scott sentence describing an ordered field such that $(\exists x)\, p(x) = 0$ and for all x and y, if $p(y) = 0$, then y is equal to some rational function of x. It follows that the index set is Π^0_2. For m-completeness, let S be a Π^0_2 set. We produce a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that if $n \in S$, then $\mathcal{A}_n \cong \mathcal{A}$, and if $n \notin S$, then $\mathcal{A}_n \cong \mathcal{A}(e)$, where $e \in \mathbb{R}$.

We need a lemma, and for later use, it is convenient to have the following definition.

Definition 4.2. Let $\mathcal{C}$ be a substructure of $\mathcal{B}$. We write

$$\mathcal{C} \leq_1 \mathcal{B}$$

if for all finitary quantifier-free formulas φ, if $\mathcal{B} \models \varphi(\overline{c}, \overline{b})$, where $\overline{c}$ is in $\mathcal{C}$, then there exists $\overline{b}'$ such that $\mathcal{C} \models \varphi(\overline{c}, \overline{b}')$.

In other words, if $\mathcal{C} \leq_1 \mathcal{B}$, then the satisfaction of finitary existential formulas by tuples in $\mathcal{C}$ is the same in $\mathcal{B}$ and $\mathcal{C}$.

Lemma 4.3. *If $\mathcal{A}$ is either $\mathbb{Q}$ or $\mathbb{Q}(a)$, where a is real algebraic, then $\mathcal{A} \leq_1 \mathcal{A}(e)$.*

Proof. Let φ be finitary quantifier-free. Suppose

$$\mathcal{A}(e) \models \varphi(\overline{c}, e, \overline{a}) \ ,$$

where $\overline{c}$ is in $\mathcal{A}$. In the case where $\mathcal{A} = \mathbb{Q}(a)$, we may suppose that $a \in \overline{c}$. Then $\overline{a}$ is in the field generated by $\overline{c}$ and e. There is an open interval I (in $\mathbb{R}$) containing e such that for all $e' \in I$, we have $\mathbb{R} \models \varphi(\overline{c}, e', \overline{a}')$, where $\overline{a}'$ is obtained from $\overline{c}, e'$ in the same way that $\overline{a}$ is obtained from $\overline{c}, e$. Taking e' to be rational, we have $\overline{a}'$ in $\mathcal{A}$ such that $\mathcal{A} \models \varphi(\overline{c}, e', \overline{a}')$.

□

Let $\mathcal{M}^+$ be a computable copy of $\mathcal{A}(e)$, and let $\mathcal{M}$ be the substructure isomorphic to $\mathcal{A}$. What is important is that $\mathcal{M}^+$ is computable, and $\mathcal{M}$ is a c.e. substructure with $\mathcal{M} \leq_1 \mathcal{M}^+$. Let B be an infinite computable set of constants, for the universe of all $\mathcal{A}_n$. We have a computable approximation $(S_s)_{s\in\omega}$ for S such that

$$n \in S \text{ iff } n \in S_s \text{ for infinitely many } s \ .$$

At each stage, we determine a finite partial $1-1$ functions f_s to the target structure, $\mathcal{M}$ if $n \in S_s$, and $\mathcal{M}^+$ otherwise. The domain of f_s includes the first s constants from B, and the range includes the first s elements of the target structure. Also, at stage s, we enumerate a finite part of D_s of the diagram of $\mathcal{A}_n$, such that f_s makes D_s true in the target structure. For the first s atomic sentences φ, involving only the first s constants, we put $\pm\varphi$ in D_s. We arrange that if $n \in S$, then $f = \cup_{n\in S_s} f_s$ is an isomorphism from $\mathcal{A}_n$ onto $\mathcal{M}_1$, and if $n \notin S$, and s_0 is least such that for all $s \geq s_0$, $n \notin S_s$, then $f = \cup_{s\geq s_0} f_s$ is an isomorphism from $\mathcal{A}_n$ onto $\mathcal{M}^+$.

At stage 0, we let $f_0 = \emptyset$, and $D_0 = \emptyset$. At stage $s+1$, if $n \in S_{s+1}$ and $n \in S_s$, or if $n \notin S_{s+1}$ and $n \notin S_s$, then we let $f_{s+1} \supseteq f_s$, adding to the domain and range. We extend D_s to D_{s+1} so that f_{s+1} makes the sentences true in the target structure. Suppose $n \in S_{s+1}$ and $n \notin S_s$. Let $t < s$ be greatest such that $n \in S_t$ or $t = 0$. We may suppose that f_s extends f_t. It follows from Lemma 4.3 that there is an extension of f_t which makes D_s true in $\mathcal{M}$. We let f_{s+1} be such an extension that also includes the required elements in the domain and range. We extend D_s to D_{s+1} so that f_{s+1} makes the sentences true. Finally, suppose $n \in S_s$ and $n \notin S_{s+1}$. We let $f_{s+1} \supseteq f_s$, changing to the larger target structure. We add the required elements to the domain and range. We extend D_s to D_{s+1} so that f_{s+1} makes the sentences true in the target structure.

Theorem 4.4. *Let K be the class of Archimedean ordered fields, and let $\mathcal{A}$ be $\mathbb{Q}(e_1, \ldots, e_k)$, where the e_i are algebraically independent reals, $k \geq 1$. Then $I(\mathcal{A})$ is m-complete d-Σ^0_2 within K.*

Proof. We have a computable Σ^0_2 sentence φ stating that there are elements $e_1^{\mathcal{A}}, \ldots, e_k^{\mathcal{A}}$ filling the cuts of $e_1, \ldots, e_k$, and we have another computable Σ^0_2 sentence ψ saying that there are elements $x_1, \ldots, x_k$ filling these cuts, and another element not equal to any rational function of them. Then φ & $\neg\psi$ is a Scott sentence for $\mathcal{A}$. It follows that $I(\mathcal{A})$ is d-Σ^0_2.

Toward completeness, let $S = S_1 - S_2$, where S_i is Σ^0_2. Let e_{k+1} be a computable real number algebraically independent of $e_1, \ldots, e_k$. Let $\mathcal{M}^+$ be a computable copy of $\mathbb{Q}(e_1, \ldots, e_k, e_{k+1})$, let $\mathcal{M}$ be the subfield isomorphic to $\mathcal{A}$, and let $\mathcal{M}^-$ be the subfield generated by the elements corresponding to e_i for $i < k$. (If $k = 1$, then $\mathcal{A}^- = \mathbb{Q}$.) We have the following lemma.

Lemma 4.5. $\mathcal{M}^- \leq_1 \mathcal{M} \leq_1 \mathcal{M}^+$

What is important is that $\mathcal{M}^+$ is a computable structure, and $\mathcal{M}^-$ and $\mathcal{M}$ are c.e. substructures such that $\mathcal{M}^- \leq_1 \mathcal{M} \leq_1 \mathcal{M}^+$. We will produce a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that if $n \notin S_1$, then $\mathcal{A}_n \cong \mathcal{M}^-$, if $n \in S_1 - S_2$, then $\mathcal{A}_n \cong \mathcal{M}$, and if $n \in S_1 \cap S_2$, then $\mathcal{A}_n \cong \mathcal{M}^+$. Let B be an infinite computable set of constants, for the universe of all $\mathcal{A}_n$. We have computable approximations $(S_{i,s})_{s\in\omega}$ for S_i such that

$$n \in S_i \text{ iff } (\exists s_0)(\forall s \geq s_0)[n \in S_{i,s}] \ .$$

At each stage s, we will have a finite partial $1-1$ function f_s from B to the appropriate target structure—$\mathcal{M}^-$ if $n \notin S_{1,s}$, $\mathcal{M}$ if $n \in S_{1,s} - S_{2,s}$, and $\mathcal{M}^+$ if $n \in S_{1,s} \cap S_{2,s}$. The domain of f_s will include the first s constants from B, and the range will include the first s elements of the target structure. We will also have a finite set D_s of atomic sentences and negations of atomic sentences such that f_s makes D_s true in the target structure. For the first s atomic sentences φ involving only the first s constants, D_s will include $\pm\varphi$. We let $\mathcal{A}_n$ be the structure with atomic diagram $\cup_s D_s$.

We will arrange that if $n \notin S_1$, then the union of f_s for $n \notin S_{1,s}$ will be an isomorphism from $\mathcal{A}_n$ onto $\mathcal{M}^-$. If $n \in S_1 - S_2$, and s_0 is first such that for all $s \geq s_0$, $n \in S_{1,s}$, then the union of f_s, for $s \geq s_0$ such that $n \notin S_{2,s}$, will be an isomorphism from $\mathcal{A}_n$ onto $\mathcal{M}$. If $n \in S_1 \cap S_2$, and s_1 is first such that for all $s \geq s_1$, $n \in S_{1,s}$ and $n \in S_{2,s}$, then the union of f_s for $s \geq s_1$ will be an isomorphism from $\mathcal{A}_n$ onto $\mathcal{M}^+$.

We start with $f_0 = \emptyset$ and $D_0 = \emptyset$. Suppose we have f_s and D_s. If the target structure at stage $s+1$ is the same as at stage s, then we extend f_s and D_s in the obvious way. Suppose $n \notin S_{1,s+1}$, where $n \in S_{1,s}$. Let $t < s$ be greatest such that $n \notin S_{1,t}$ or $t = 0$. Say f_t maps $\overline{d}$ to $\overline{c}$ in $\mathcal{M}^-$. We let $f_{s+1} \supseteq f_t$, where f_{s+1} makes the sentences of D_s true in $\mathcal{M}^-$. Now, suppose $n \in S_{1,s+1} - S_{2,s+1}$, where $n \in S_{1,s} \cap S_{2,s}$. Take $t < s$ greatest such that $n \in S_{1,t} - S_{2,t}$ and for all $t < t' < s$, $n \in S_{1,t'}$. Then we let $f_{s+1} \supseteq f_t$, where f_{s+1} makes the sentences of D_s true in $\mathcal{M}$.

Suppose that $n \in S_{1,s+1} - S_{2,s+1}$, where $n \notin S_{1,s}$. Then we take $f_{s+1} \supseteq f_s$, where f_{s+1} makes D_s true in the target structure $\mathcal{M}$. Similarly, if we have $n \in S_{1,s+1} \cap S_{2,s+1}$, where either $n \notin S_{1,s}$ or $n \in S_{1,s} - S_{2,s}$, we let $f_{s+1} \supseteq f_s$, where f_{s+1} makes D_s true in the target structure $\mathcal{M}^+$. We extend f_s to include the required elements in the domain and range, and we let $D_{s+1} \supseteq D_s$ so that f_{s+1} makes D_{s+1} true.

□

Next, we consider Archimedean ordered fields with infinite transcendence degree. As for vector spaces, the complexity goes up by one quantifier. However, the proof is different. For an infinite dimensional vector space, we had proper substructures isomorphic to the whole, while for an Archimedean ordered field, this does not happen. (See [2] for a similar proof.)

Theorem 4.6. *Let K be the class of Archimedean ordered fields, and let $\mathcal{A} \in K$ be the least field generated by a family of infinitely many algebraically independent reals. Then $I(\mathcal{A})$ is m-complete Π^0_3 within K.*

Proof. We shall describe a computable Π_3 Scott sentence for $\mathcal{A}$. For each $a \in \mathcal{A}$, let $c_a(x)$ be the conjunction of formulas saying $q < x < q'$, where q, q' are rationals, and $\mathcal{A} \models q < a < q'$. The Scott sentence describes an Archimedean ordered field such that

$$\bigwedge_{a \in \mathcal{A}} (\exists x)\, c_a(x) \;\&\; (\forall x) \bigvee_{a \in \mathcal{A}} c_a(x) \ .$$

For completeness, we need the following lemma.

Lemma 4.7. *There is a sequence $(\mathcal{M}_k)_{k\in\omega}$ of subfields of $\mathcal{A}$, uniformly c.e., such that for all k, $\mathcal{M}_{k+1}$ properly extends $\mathcal{M}_k$, and $\cup_k \mathcal{M}_k = \mathcal{A}$.*

Proof. Let $(a_n)_{n\in\omega}$ be a computable list of all elements of $\mathcal{A}$. We can arrange that $\mathcal{M}_0$ is the prime field $\mathcal{A}$, and $\mathcal{M}_{k+1}$ is field generated by $\mathcal{M}_k$ and a_{n_k}, where n_k is first such that $a_{n_k} \notin \mathcal{M}_k$.

□

We need the following property.

Lemma 4.8. *For all k, $\mathcal{M}_k \leq_1 \mathcal{M}_{k+1}$. Hence, if $k < m$, then $\mathcal{M}_k \leq_1 \mathcal{M}_m$.*

Proof. Suppose $\mathcal{M}_{k+1} \models \varphi(\overline{c}, \overline{b})$, where φ is quantifier-free, and $\overline{c}$ is in $\mathcal{M}_k$. Let n_k be least such that $a_{n_k} \notin \mathcal{M}_k$. We may suppose that $\overline{c}$ includes a_m, for all $m < n_k$, and $\overline{b}$ has the form $a_{n_k}, \overline{d}$. Say $\psi(\overline{c}, a_{n_k}, \overline{d})$ says how $\overline{d}$ are expressed as rational functions of $\overline{c}, a_{n_k}$. There is an interval I around a_{n_k}, with rational endpoints, such that for all $x \in I$, and all $\overline{u}$ satisfying $\psi(\overline{c}, x, \overline{u})$, $\varphi(\overline{c}, x, \overline{u})$ holds in $\mathbb{R}$. Taking x rational, we have $\overline{u}$ satisfying $\psi(\overline{c}, x, \overline{u})$ in $\mathcal{M}_k$, so $\mathcal{M}_k \models \varphi(\overline{c}, x, \overline{u})$.

□

Recall that $Cof = \{n : \omega - W_n \text{ is finite}\}$. This set is m-complete Σ_3^0, so the complement is m-complete Π_3^0. We have a Δ_2^0 function $\nu(n,r)$, non-decreasing in r, for each n, such that if $n \in Cof$, then $\lim_r \nu(n,r)$ has value equal to the cardinality of $\omega - W_n$, and if $n \notin Cof$, then $\lim_r \nu(n,r) = \infty$. We have a computable approximation to ν, and we define a computable function $g(n,s)$, such that $g(n,0)$ is our stage 0 guess at $\nu(n,0)$. Supposing that $g(n,s)$ is our stage s guess at $\nu(n,r)$, if our stage $s+1$ guess at ν agrees with the stage s guess at $\nu(n,j)$ for all $j \leq r$, then $g(n,s+1)$ is the stage $s+1$ guess at $\nu(n,r+1)$. If there is a disagreement, say j is least such that our stage s and stage $s+1$ guesses at $\nu(n,j)$ disagree. Let $g(n,s+1)$ be the stage $s+1$ guess at $\nu(n,j)$. If $n \notin Cof$, then for all k, there exists s_0 such that

$$(\forall s \geq s_0)\,[g(n,s) \geq k] \ .$$

If $n \in Cof$, and k is the cardinality of $\omega - W_n$, then there are infinitely many s such that $g(n,s) = k$, and k is the least such number. In other words, $\liminf_s g(n,s)$ is the cardinality of $\omega - W_n$.

We will build a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of Archimedean ordered fields such that if $n \notin Cof$, that is, $\omega - W_n$ is infinite, then $\mathcal{A}_n \cong \mathcal{A}$. If $n \in Cof$, that is, $\omega - W_n$ has size k for some $k \in \omega$, then $\mathcal{A}_n \cong \mathcal{M}_k$. Let B be an infinite computable set of constants, for the universe of all $\mathcal{A}_n$. At stage s, we have a finite partial $1-1$ function f_s from B to the target structure $\mathcal{M}_k$, where $k = g(n,s)$. The domain of f_s will include the first s constants from B, and the range will include the first s elements of the target structure. We also have D_s, a finite set of atomic sentences and negations of atomic sentences such that f_s makes D_s true in the target structure. For the first s atomic sentences φ using only the first s constants, D_s will include $\pm\varphi$.

We let $\mathcal{A}_n$ be the structure with

$$D(\mathcal{A}_n) = \cup_s D_s \ .$$

Let T be the set of s such that for all $t \geq s$, $g(n,t) \geq g(n,s)$. We shall arrange that $f = \cup_{s\in T} f_s$ is an isomorphism from $\mathcal{A}_n$ onto the desired structure. With this goal, we maintain the following condition.

Condition Maintained: Suppose $t < s$, where $g(n,t) \leq g(n,s)$, and for all $t < t' < s$, we have $g(n,t') \geq g(n,t)$. Then $f_s \supseteq f_t$.

Let $f_0 = \emptyset$, and let $D_0 = \emptyset$. Given f_s and D_s, we must determine f_{s+1} and D_{s+1}. The target structure is $\mathcal{M}_k$, where $k = g(n,s+1)$. First, suppose $g(n,s+1) \geq g(n,s)$. Then we let $f_{s+1} \supseteq f_s$. We extend to include the required elements in the domain and range. We let $D_{s+1} \supseteq D_s$, where f_{s+1} makes the sentences true in the target structure. Now, suppose $g(n,s+1) < g(n,s)$. Let $t < s$ be greatest such that $g(n,t) = g(n,s+1)$ and there is no $t < t' < s$ such that $g(n,t') < g(n,t)$, or if there is no $t < s$ such that $g(n,t) = g(n,s+1)$, take the greatest $t < s$ such that $g(n,t) \leq g(n,s+1)$. We may assume that f_s

extends f_t. We let f_{s+1} extend f_t such that f_{s+1} makes D_s true in the target structure.

□

What is important in the proof above is that $\mathcal{A}$ is computable, the substructures $\mathcal{M}_k$ are computably ennumerable, uniformly in k, and they form a $\leq_1$-elementary chain with union $\mathcal{A}$. For an arbitrary computable Archimedean ordered field of infinite transcendence degree, we would have trouble getting the $\leq_1$-elementary chain. The difficulties disappear when we consider real closed fields.

Theorem 4.9. *Let K be the class of computable Archimedean real closed ordered fields, and let $\mathcal{A}$ be a member of K.*

1. *If the transcendence degree of $\mathcal{A}$ is 0, then $I(\mathcal{A})$ is m-complete Π^0_2 within K.*
2. *If the transcendence degree of $\mathcal{A}$ is finite but greater than 0, then $I(\mathcal{A})$ is m-complete d-Σ^0_2 within K.*
3. *If the transcendence degree of $\mathcal{A}$ is infinite then $I(\mathcal{A})$ is m-complete Π^0_3 within K.*

Proof. For 1, we have a computable Π_2 Scott sentence describing the real closed ordered fields in which every element is a root of some non-trivial polynomial with integer coefficients. It follows that $I(\mathcal{A})$ is Π^0_2. For completeness, the proof is as for $\mathbb{Q}$ or $\mathbb{Q}(a)$, where a is real algebraic. Let S be a Π^0_2 set. We can produce a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that if $n \in S$, then $\mathcal{A}_n \cong \mathcal{A}$, and otherwise, $\mathcal{A}_n \cong \mathcal{A}(e)$. Let $\mathcal{M}^+$ be a computable copy of $\mathcal{A}(e)$, and let $\mathcal{M}$ be the substructure isomorphic to $\mathcal{A}$. Note that $\mathcal{M}$ is c.e., and it is an elementary substructure of $\mathcal{M}^+$.

For 2, let $\mathcal{A}$ be the real closure of a set of $e_1, \ldots, e_k$, where the e_i are algebraically independent computable reals. We have a Scott sentence describing a real closed Archimedean ordered field, with elements filling the cuts of the chosen e_i and not having $k+1$ algebraically independent elements. This sentence can be put in the form $\varphi \ \& \ \neg\psi$, where φ and ψ are computable Σ_2. It follows that $I(\mathcal{A})$ is d-Σ^0_2.

For completeness, the proof is the same as for $\mathbb{Q}(e_0, \ldots, e_{k-1})$. Let e_k be a further computable real, independent of $e_0, \ldots, e_{k-1}$. Let $\mathcal{M}^+$ be a computable real closed ordered field, with elements filling the cuts of e_i for $i \leq k$. Let $\mathcal{M}$ be the subfield isomorphic to $\mathcal{A}$, and let $\mathcal{M}^-$ be the real closure of the elements corresponding to e_i, for $i < k-1$. Then $\mathcal{M}^-$ and $\mathcal{M}$ are c.e. substructures of $\mathcal{M}^+$, and

$$\mathcal{M}^- \leq_1 \mathcal{M} \leq_1 \mathcal{M}^+ .$$

(In fact, we have elementary substructures here.) We can produce a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that if $n \notin S_1$, then $\mathcal{A}_n \cong \mathcal{M}^-$, if $n \in S_1 - S_2$, then $\mathcal{A}_n \cong M$, and if $n \in S_1 \cap S_2$, then $\mathcal{A}_n \cong \mathcal{M}^+$.

For 3, we have a computable Π_3 Scott sentence describing an Archimedean real closed ordered field in which the cuts filled are exactly those of the elements of $\mathcal{A}$. For completeness, the proof is as for $\mathbb{Q}(e_1, e_2, \ldots)$, where $e_1, e_2, \ldots$ are algebraically independent reals. We have $\mathcal{A}$ computable. Let $(a_i)_{i\in\omega}$ be a computable list of all elements of $\mathcal{A}$, and let $\mathcal{M}_k$ be the real closure of $\{a_i : i < k\}$ in $\mathcal{A}$. Then the structures $\mathcal{M}_k$ are c.e. uniformly in k, and they form an elementary chain with union $\mathcal{A}$. Therefore, we can produce a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that if $n \notin Cof$, then $\mathcal{A}_n \cong \mathcal{A}$, and if $n \in Cof$, and k is the cardinality of $\omega - W_n$, then $\mathcal{A}_n \cong \mathcal{M}_k$.

□

5 Abelian p-Groups

5.1 Preliminaries on Abelian p-Groups

Fix a prime p. An Abelian group G is a *p-group* if for each $x \in G$, the order of x is p^n for some n. We will consider only countable Abelian p-groups. These groups are of particular interest because of their classification up to isomorphism by Ulm. For a classical discussion of this theorem and a more detailed discussion of this class of groups, consult Kaplansky's book [10]. Generally, notation here will be similar to Kaplansky's.

Let G be a countable Abelian p-group. We define a sequence of subgroups G_α, letting $G_0 = G$, $G_{\alpha+1} = pG_\alpha$, and for limit α, $G_\alpha = \cap_{\beta<\alpha} G_\beta$. There is a countable ordinal α such that $G_\alpha = G_{\alpha+1}$. The least such α is the *length* of G, denoted by $\lambda(G)$. The group is *reduced* if $G_{\lambda(G)} = \{0\}$. An element $x \neq 0$ has *height* β if $x \in G_\beta - G_{\beta+1}$. Let $P(G)$ be the set of element of G of order p. Let $P_\alpha = G_\alpha \cap P(G)$. For each $\beta < \lambda$, $P_\beta / P_{\beta+1}$ is a vector space over $\mathbb{Z}_p$ of dimension $\leq \aleph_0$, with dimension denoted by $u_\beta(G)$. The *Ulm sequence* for G is the sequence $(u_\beta(G))_{\beta<\lambda(G)}$.

For any computable ordinal α, it is somewhat straightforward to write a computable infinitary sentence stating that G is a reduced Abelian p-group of length at most α and describing its Ulm invariants. In particular, Barker [1] verified the following.

Lemma 5.1. *Let G be a computable Abelian p-group.*

1. $G_{\omega\cdot\alpha}$ *is* $\Pi^0_{2\alpha}$.
2. $G_{\omega\cdot\alpha+m}$ *is* $\Sigma^0_{2\alpha+1}$.
3. $P_{\omega\cdot\alpha}$ *is* $\Pi^0_{2\alpha}$.
4. $P_{\omega\cdot\alpha+m}$ *is* $\Sigma^0_{2\alpha+1}$.

Proof. It is easy to see that 3 and 4 follow from 1 and 2 respectively. Toward 1 and 2, note the following:

$$
\begin{aligned}
x \in G_m &\iff \exists y(p^m y = x)\ ; \\
x \in G_\omega &\iff \bigwedge\!\!\!\bigwedge_{m\in\omega} \exists y(p^m y = x)\ ; \\
x \in G_{\omega\cdot\alpha+m} &\iff \exists y[p^m y = x \ \&\ G_{\omega\cdot\alpha}(y)]\ ; \\
x \in G_{\omega\cdot\alpha+\omega} &\iff \bigwedge\!\!\!\bigwedge_{m\in\omega} \exists y[p^m y = x \ \&\ G_{\omega\cdot\alpha}(y)]\ ; \\
x \in G_{\omega\cdot\alpha} &\iff \bigwedge\!\!\!\bigwedge_{\gamma<\alpha} G_{\omega\cdot\gamma}(x) \text{ for limit } \alpha\ .
\end{aligned}
$$

□

Using Lemma 5.1, it is easy to write, for any computable ordinal β, a computable $\Pi^0_{2\beta+1}$ sentence whose models are exactly the reduced Abelian p-groups of length $\omega\beta$.

Khisamiev gave a useful characterization of those Abelian p-groups which have computable copies [13], [12], at least for certain lengths. For groups of finite length, it is easy to produce computable copies. Khisamiev gave a characterization for length ω, and proved an inductive lemma that allowed him to build up to all lengths less than ω^2. Here is the result for length ω.

Proposition 5.2 (Khisamiev). *Let $\mathcal{A}$ be a reduced Abelian p-group of length ω. Then $\mathcal{A}$ has a computable copy iff*

1. *the relation $R_{\mathcal{A}} = \{(n,k) : u_n(\mathcal{A}) \geq k\}$ is Σ^0_2, and*
2. *there is a computable function $f_{\mathcal{A}}$ such that for each n, $f_{\mathcal{A}}(n,s)$ is non-decreasing, with limit $n^* \geq n$ such that $u_{n^*}(\mathcal{A}) \neq 0$.*

Moreover, we can effectively determine a computable index for a copy of $\mathcal{A}$ from a Σ^0_2 index for $R_{\mathcal{A}}$ and a computable index for a function $f_{\mathcal{A}}$.

Here is the inductive lemma.

Lemma 5.3 (Khisamiev). *Let $\mathcal{A}$ be a reduced Abelian p-group. Suppose $\mathcal{A}_\omega$ is Δ^0_3, the relation $R_{\mathcal{A}}$ is $\Sigma^0_2(X)$, and there is a function $f_{\mathcal{A}}$ such that for all n, $f_{\mathcal{A}}(n,s)$ is non-decreasing, with limit $r^* \geq r$ such that $u_{r^*}(\mathcal{A}) \neq 0$. Then $\mathcal{A}$ has a computable copy, with index computed effectively from those for $\mathcal{A}_\omega$, $R_{\mathcal{A}}$, and $f_{\mathcal{A}}$.*

The results above relativize. They yield the following two theorems, which we shall use in what follows.

Theorem 5.4 (Khisamiev). *Let $\mathcal{A}$ be a reduced Abelian p-group of length ωM, where $M \in \omega$. Then $\mathcal{A}$ has a computable copy iff for each $k < M$,*

1. *the relation $R^k_{\mathcal{A}} = \{(r,t) | u_{\omega k + r}(\mathcal{A}) \geq t\}\}$ is Σ^0_{2k+2}, and*

2. *there is a* Δ^0_{2k+1} *function* $f^k_{\mathcal{A}}(r,s)$ *such that for each* n, *the function* $f^k_{\mathcal{A}}(n,s)$ *is nondecreasing, with limit* $r^* \geq r$ *such that* $u_{\omega k+r^*}(\mathcal{A}) \neq 0$.

Moreover, we can pass effectively from Σ^0_{2k+2} *indices for the relations* $R^k_{\mathcal{A}}$ *and* Δ^0_{2k+1} *indices for appropriate functions* $f^k_{\mathcal{A}}$ *to a computable index for a copy of* $\mathcal{A}$.

Theorem 5.5 (Khisamiev). *Let* $\mathcal{A}$ *be a reduced Abelian* p*-group of length* $\leq \omega M$. *Suppose* $\mathcal{A}_{\omega M}$ *is* Δ^0_{2M+1}, *and for each* $k < M$,

1. *the relation* $R^k_{\mathcal{A}} = \{(r,t) | u_{\omega k+r}(\mathcal{A}) \geq t\}\}$ *is* Σ^0_{2k+2}, *and*

2. *there is a* Δ^0_{2k+1} *function* $f^k_{\mathcal{A}}(r,s)$ *such that for each* n, $f^k_{\mathcal{A}}(n,s)$ *is nondecreasing, with limit* $r^* \geq r$ *such that* $u_{\omega k+r^*}(\mathcal{A}) \neq 0$.

Then $\mathcal{A}$ *has a computable copy, with index computed effectively from the* Δ^0_{2M+1} *index for* $\mathcal{A}_{\omega M}$, Σ^0_{2k+2} *indices for* $R^k_{\mathcal{A}}$ *and* Δ^0_{2k+1} *indices for appropriate functions* $f^k_{\mathcal{A}}$.

5.2 Index Sets of Groups of Small Ulm Length

In [3], it is shown that for the countable reduced Abelian p-group of length ωM with uniformly infinite Ulm invariants, the index set is m-complete Π^0_{2M+1}. It seemed that other complexities might be possible for groups of the same length. However, it turned out that the index set for any group of length ωM is also m-complete Π^0_{2m+1}. The case $M = 1$ of the following theorem was first proved in [4].

Proposition 5.6. *Let* K *be the class of reduced Abelian* p*-groups of length* ωM, *and let* $\mathcal{A} \in K$. *Then* $I(\mathcal{A})$ *is* m*-complete* Π^0_{2M+1} *within* K.

Proof. Let $\mathcal{A} \in K$. First, we show that $\mathcal{A}$ has a computable Π_{2M+1} Scott sentence. There is a computable Π_2 sentence θ characterizing the Abelian p-groups. Next, there is a computable Π_{2M+1} sentence λ characterizing the groups which are reduced and have length at most ωM. For each $\alpha < \omega M$, we can find a computable Σ_{2M} sentence $\varphi_{\alpha,k}$ saying that $u_\alpha \geq k$. The set of these Σ_{2M} sentences true in $\mathcal{A}$ is Σ^0_{2M}. For each $\varphi_{\alpha,k}$, we can find a computable Π_{2M} sentence equivalent to the negation, and the set of these sentences true in $\mathcal{A}$ is Π^0_{2M}. We have a computable Π_{2M+1} sentence υ equivalent to the conjunction of the sentences $\pm\varphi_{\alpha,k}$ true in $\mathcal{A}$. Then we have a computable Π_{2M+1} Scott sentence equivalent to θ & λ & υ. It follows that $I(\mathcal{A})$ is Π^0_{2M+1}.

For completeness, let S be a Π^0_{2M+1} set. We will produce a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of elements of K, such that $n \in S$ if and only if $\mathcal{A}_n \cong \mathcal{A}$. We will specify $\mathcal{A}_n$ by giving relations $R^k_{\mathcal{A}_n}$ and functions $f^k_{\mathcal{A}_n}$, for $k < M$, as in Theorem 5.4. Since $\mathcal{A}$ is computable, there are relations $R^k_{\mathcal{A}}$ and functions $f^k_{\mathcal{A}}$, as in the theorem. For $k < M - 1$, we let $R^k_{\mathcal{A}_n} = R^k_{\mathcal{A}}$ and $f^k_{\mathcal{A}_n} = f^k_{\mathcal{A}}$.

We will define a single Δ^0_{2M-1} function g to serve as $f^{M-1}_{\mathcal{A}_n}$ for all n. We then define a family of relations R_n to serve as $R^{M-1}_{\mathcal{A}_n}$, with the feature that if $(m, k+1)$ is included, so is (m, k). The relations will be uniformly Σ^0_{2M}. Moreover, if $n \in S$, then we will have $R_n = R^{M-1}_{\mathcal{A}}$, and if $n \notin S$, then R_n will include the pairs $(m, 1)$ for $m = \lim_s g(r, s)$, but will omit some of the other pairs $(m, 1)$ in $R_{\mathcal{A}}$. Having determined g and R_n for $n \in \omega$, we will be in a position to apply Theorem 5.4.

To get the function g, we first define a Δ^0_{2M} sequence $(k_n)_{n\in\omega}$, where

$$k_0 = \lim_s f^{M-1}_{\mathcal{A}}(0, s) \text{ and } k_{m+1} = \lim_s f^{M-1}_{\mathcal{A}}(k_{m+1}, s) .$$

We will define $g(r, s)$ in such a way that $\lim_s g(r, s) = k_{2r+1}$. We have a Δ^0_{2M-1} approximation to the sequence $(k_n)_{n\in\omega}$. Let $k_{m,0} = m$ for all m, let $k_{0,s+1} = f^{M-1}_{\mathcal{A}}(0, s+1)$, and let $k_{m+1,s+1}$ be the maximum of $f^{M-1}_{\mathcal{A}}(k_{m+1,s+1}, s+1)$ and $k_{m+1,s}$. For each m, the sequence $k_{m,s}$ is nondecreasing in s and has limit k_m. Define $g(r, s) = k_{2r+1,s}$.

The relation $R^{M-1}_{\mathcal{A}}$ is c.e. in Δ^0_{2M}, and we have a sequence of finite approximations $R^{M-1}_{\mathcal{A},s}$. The set S is Π^0_1 over Δ^0_{2M}. We have a Δ^0_{2M} approximation $(S_s)_{s\in\omega}$ such that if $n \in S$, then $n \in S_s$, for all s, and if $n \notin S$, then there exists s_0 such that for $s < s_0$, $n \in S_s$, and for $s \geq s_0$, $n \notin S_s$. We define uniformly Σ^0_{2M} relations R_n. At stage 0, we have $R_{n,0} = \emptyset$. At stage $s+1$, we extend $R_{n,s}$ to $R_{n,s+1}$. We add the pair $(k_{2s+1}, 1)$. If $n \notin S_s$, this is all, but if $n \in S_s$, we include all pairs (r, k) in $R^{M-1}_{\mathcal{A},s}$. If $n \in S$, then $R_n = R^{M-1}_{\mathcal{A}}$. If $n \notin S$, then R_n includes only finitely many of the pairs $(k_{2s}, 1)$, while $R^{M-1}_{\mathcal{A}}$ includes all of them.

We apply Theorem 5.4 as planned to obtain a uniformly computable sequence of reduced Abelian p-groups $(\mathcal{A}_n)_{n\in\omega}$, all of length ωM, such that $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$.

□

We have considered groups of limit length ωM. Now, we consider groups of successor length. There are several cases.

Proposition 5.7. *Let K be the class of reduced Abelian p-groups of length $\omega M + N$, where $N > 0$. Suppose $\mathcal{A}$ is a computable member of K, where $\mathcal{A}_{\omega M}$ is finite.*

1. *If $\mathcal{A}_{\omega M}$ is minimal for the prescribed length, (i.e. if it is of type $\mathbb{Z}_{p^N}$), then $I(\mathcal{A})$ is m-complete Π^0_{2M+1} within K. (It is m-complete d-Σ^0_{2M+1} within the class of groups of length $\leq N$.)*
2. *If $\mathcal{A}_{\omega M}$ is not minimal, then $I(\mathcal{A})$ is m-complete d-Σ^0_{2M+1} within K.*

Proof. We use the following lemma.

Lemma 5.8. *Let $\mathcal{C}$ be a non-trivial finite Abelian p-group of length N.*

1. *If $\mathcal{C}$ is minimal among groups of length N, of type $\mathbb{Z}_{p^N}$, then it has a finitary Π_1 Scott sentence within groups of length N. For any X, and any set S that is $\Pi^0_1(X)$, there is a uniformly X-computable sequence $(\mathcal{C}_n)_{n\in\omega}$ consisting of groups of length N such that*

$$\mathcal{C}_n \cong \mathcal{C} \text{ iff } n \in S \ .$$

2. *If $\mathcal{C}$ is not minimal among groups of length N, then it has a finitary d-c.e. Scott sentence, and for any X, and any set S that is d-c.e. relative to X, there is a uniformly X-computable sequence $(\mathcal{C}_n)_{n\in\omega}$ consisting of groups of length N such that*

$$\mathcal{C}_n \cong \mathcal{C} \text{ iff } n \in S \ .$$

Proof. For 1, we have a finitary Π_1 Scott sentence within K saying that there are no more than p^N elements. The construction will be uniform in X, so without loss of generality we assume that $X = \emptyset$. If S is Π^0_1, we have a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ of Abelian p-groups, all of length N, such that if $n \in S$, then $\mathcal{C}_n \cong \mathbb{Z}_{p^N}$, and if $n \notin S$, then $\mathcal{C}_n \cong \mathbb{Z}^2_{p^N}$.

For 2, we have a finitary d-c.e. Scott sentence, as in the section on finite structures. Suppose $S = S_1 - S_2$, where S_1 and S_2 are computably enumerable. We let $\mathcal{C}^-$ be a proper subgroup of $\mathcal{C}$, still of length N, and we let $\mathcal{C}^+$ be a proper extension of $\mathcal{C}$, also of length N. We get a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ such that $\mathcal{C}_n \cong \mathcal{C}^-$ if $n \notin S_1$, $\mathcal{C}_n \cong \mathcal{C}$ if $n \in S_1 - S_2$, and $\mathcal{C}_n \cong \mathcal{C}^+$ if $n \in S_1 \cap S_2$.

□

Now, we can prove the proposition. Let $\mathcal{C} = \mathcal{A}_{\omega M}$. For 1, we have a computable Π_{2M+1} sentence characterizing the groups $\mathcal{G}$ such that $\mathcal{G}_{\omega M} \cong \mathcal{C}$ within the class of reduced Abelian p-groups of length $\omega M + N$. We have a computable Π_{2M+1} sentence characterizing the Abelian p-groups $\mathcal{G}$ such that for all $\alpha < \omega M$, $u_\alpha(\mathcal{G}) = u_\alpha(\mathcal{A})$. The conjunction, equivalent to a computable Π_{2M+1} sentence, is a Scott sentence for $\mathcal{A}$.

For completeness, let S be Π^0_{2M+1}. Note that S is Π^0_1 over Δ^0_{2M+1}. By Lemma 5.8, we have a uniformly Δ^0_{2M+1} sequence $(\mathcal{C}_n)_{n\in\omega}$ of groups of length N such that $\mathcal{C}_n \cong \mathcal{C}$ iff $n \in S$. Since $\mathcal{A}$ is computable, we get Σ^0_{2k+2} relations $R^k_{\mathcal{A}}$, and Δ^0_{2k+1} functions f^k, as required in Theorem 5.5. We obtain a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of groups of length $\omega M + N$ such that $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$.

For 2, we have a computable d-Σ_{2M+1} sentence characterizing the groups $\mathcal{G}$ such that $\mathcal{G}_{\omega M} \cong \mathcal{C}$. We have a computable Π_{2M+1} sentence characterizing the Abelian p-groups such that for all $\alpha < \omega M$, $u_\alpha(\mathcal{G}) = u_\alpha(\mathcal{A})$. The conjunction, equivalent to a computable d-Σ_{2M+1} sentence, is a Scott sentence for $\mathcal{A}$.

Toward completeness, let S be a d-Σ^0_{2M+1} set. Then S is d-c.e. relative to Δ^0_{2M+1}. By Lemma 5.8, there is a uniformly Δ^0_{2M+1} sequence $(\mathcal{C}_n)_{n\in\omega}$ of groups of length N such that $\mathcal{C}_n \cong \mathcal{C}$ iff $n \in S$. As above, since $\mathcal{A}$ is computable, we have relations $R^k_{\mathcal{A}}$ and functions f^k for $k < M$, as required in Theorem 5.5. We get a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of groups of length $\omega M + N$ such that $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$.

□

We continue with reduced Abelian p-groups $\mathcal{A}$ of length $\omega M + N$, but now we suppose that $\mathcal{A}_{\omega M}$ is infinite. This means that for some $k < N$, $u_{\omega M+k}(\mathcal{A}) = \infty$.

Proposition 5.9. *Let K be the class of reduced Abelian p-groups of length $\omega M + N$. Let $\mathcal{A}$ be a computable member of K. If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{A}) = \infty$, and for all $m < k$ we have $u_{\omega M+m}(\mathcal{A}) = 0$, then $I(\mathcal{A})$ is m-complete Π^0_{2M+2} within K.*

Proof. We use the following lemma.

Lemma 5.10. *Suppose $\mathcal{C}$ is a reduced Abelian p-group of length N, where there is a unique $k < N$ such that $u_k(\mathcal{C}) = \infty$, and for all $m < k$, $u_m(\mathcal{C}) = 0$.*

1. *The structure $\mathcal{C}$ has a computable Π_2 Scott sentence.*
2. *For any set X, if S is $\Pi^0_2(X)$, there is a uniformly X-computable sequence $(\mathcal{C}_2)_{n\in\omega}$ of reduced Abelian p-groups, all of length N, such that*

$$\mathcal{C}_n \cong \mathcal{C} \textit{ iff } n \in S .$$

Proof. We have $\mathcal{C} \cong H \oplus \mathbb{Z}^{\infty}_{p^{k+1}}$, where H is a finite direct sum of $\mathbb{Z}_{p^{i+1}}$ for $k < i < N$. Even part 1 requires some effort. There is a computable Π_2 sentence characterizing the Abelian p-groups. There is a computable Π_1 sentence saying that the length is at most N. There is a computable Π_2 sentence saying that $u_m = 0$ for all $m < k$. There is a finitary d-Σ_1 sentence characterizing the groups $\mathcal{G}$ of length N such that $\mathcal{G}_{k+1} \cong \mathcal{C}_{k+1}$. Finally, there is a computable Π_2 sentence saying that, for all r, there exists a substructure of type $\mathbb{Z}^r_{p^{k+1}}$. The conjunction of these is equivalent to a computable Π_2 sentence. We show that it is a Scott sentence for $\mathcal{C}$.

We show that if $\mathcal{G}$ is a model of the proposed Scott sentence, then $u_k(\mathcal{G}) = \infty$. To show that $u_k(\mathcal{G}) \geq m$, consider the set of statements $z_1x_1 + \ldots + z_mx_m = h$, where $z_i \in \mathbb{Z}_p$ and $h \in \mathcal{G}_{k+1}$. Say the number of these statements is r. By Ramsey's Theorem (the finite version), there exists M such that

$$M \to (2m)^k_r \ ;$$

that is, for any partition of k-sized subsets of a set of size M into r classes, there is a set of size $2m$ which is "homogeneous" in the sense that all k-sized subsets lie in the same class in the partition (for example, see [9]). Take a substructure of $\mathcal{G}$ of type $\mathbb{Z}^M_{p^{k+1}}$, and from each factor $\mathbb{Z}_{p^{k+1}}$, take an element b_i of height k and order p. If there is no m-sized subset independent over $\mathcal{G}_{k+1}$, then for each m-sized subset $b_{i_1}, \ldots, b_{i_m}$ (with $i_1 < \ldots < i_m$), one of the r statements above is satisfied. We partition according to the first such statement. Take a homogeneous set of size $2m$, with all m-tuples satisfying the statement $z_1x_1 + \ldots + z_nx_n = h$. We have disjoint m-tuples $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$ such that

$$z_1c_1 + \ldots + z_mc_m = z_1d_1 + \ldots + z_md_m \ .$$

This is impossible. Therefore, we have $u_k(\mathcal{G}) \geq m$.

For 2, assume without loss of generality that $X = \emptyset$. Let S be a Π^0_2 set. We can produce a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ such that if $n \in S$, then $\mathcal{C}_n \cong H \oplus \mathbb{Z}^\infty_{p^{k+1}}$, and otherwise $\mathcal{C}_n \cong H \oplus \mathbb{Z}^r_{p^{k+1}}$, for some finite r. We start with a copy of H. At each stage when we believe $n \in S$, we add a new direct summand of type $\mathbb{Z}_{p^{k+1}}$. Otherwise, we add nothing. The resulting sequence has the desired properties.

□

Using Lemma 5.10, we can prove the proposition. Let $\mathcal{C} = \mathcal{A}_{\omega M}$. We have a computable d-c.e. Scott sentence for $\mathcal{C}$. It follows that there is a computable d-Σ_{2M+1} sentence describing the groups $\mathcal{G}$ with $\mathcal{G}_{\omega M} \cong \mathcal{C}$. We have a computable Π_{2m+1} sentence characterizing the Abelian p-groups $\mathcal{G}$ such that for $\alpha < \omega M$, $u_\alpha(\mathcal{G}) = u_\alpha(\mathcal{A})$. There is a computable Π_{2M+2} sentence equivalent to the conjunction, and this is a Scott sentence for $\mathcal{A}$.

For completeness, let $\mathcal{C} = \mathcal{A}_{\omega M}$. Let S be Π^0_{2M+2}. Then S is Π^0_2 over Δ^0_{2M+1}. By Lemma 5.10, we get a sequence $(\mathcal{C}_n)_{n\in\omega}$, uniformly Δ^0_{2M+1}, such that for all n, $\mathcal{C}_n$ has length N, and $\mathcal{C}_n \cong \mathcal{C}$ iff $n \in S$. Now, we apply Theorem 5.5. Since $\mathcal{A}$ is computable, we have Σ^0_{2k+2} relations $R^k_\mathcal{A}$ and Δ^0_{2k+1} functions $f^k_\mathcal{A}$ for all $k < M$. From these, together with the sequence $(\mathcal{C}_n)_{n\in\omega}$, we obtain a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that for all n, $\mathcal{A}_n$ has length $\omega M + N$, and $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$.

□

Proposition 5.11. *Let K be the class of reduced Abelian p-groups of length $\omega M + N$ for some $M, N \in \omega$. Let $\mathcal{A} \in K$. If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{A}) = \infty$, and for at least one $m < k$ we have $0 < u_{\omega M+m}(\mathcal{A}) < \infty$, then $I(\mathcal{A})$ is m-complete d-Σ^0_{2M+2} within K.*

Proof. We use the following lemma.

Lemma 5.12. *Suppose $\mathcal{C}$ has length N. Suppose there is a unique $k < N$ such that $u_k(\mathcal{C}) = \infty$, and for at least one $m < k$, $0 < u_m(\mathcal{C}) < \infty$.*

1. *The structure $\mathcal{C}$ has a computable d-Σ_2 Scott sentence.*

2. *For any X, if S is d-$\Sigma^0_2(X)$, there is a uniformly X-computable sequence $(\mathcal{C}_n)_{n\in\omega}$ of reduced Abelian p-groups of length N such that*

$$\mathcal{C}_n \cong \mathcal{C} \quad \textit{iff } n \in S \ .$$

For 1, the Scott sentence is the same as in Lemma 5.10, except that we must specify u_m, for $m < k$. If $u_m \neq 0$, we need a computable d-Σ_2 sentence.

For 2, assume without loss of generality that $X = \emptyset$. Let S be d-Σ^0_2, say $S = S_1 - S_2$, where S_i is Σ^0_2. Let $\mathcal{C}^-$ have the same Ulm sequence as $\mathcal{C}$ except that $u_m(\mathcal{C}^-) = 0$. Let $\mathcal{C}^+$ have the same Ulm sequence as $\mathcal{C}$ except that $u_m(\mathcal{C}^+) > u_m(\mathcal{C})$. We produce a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ of Abelian p-groups of length N such that if $n \notin S_1$, then $\mathcal{C}_n \cong \mathcal{C}^-$, if $n \in S_1 - S_2$, then $\mathcal{C}_n \cong \mathcal{C}$, and if $n \in S_1 \cap S_2$, then $\mathcal{C}_n \cong \mathcal{C}^+$.

We start with H and add further direct summands $\mathbb{Z}_{p^{i+1}}$. At stage s, if we believe $n \notin S_1$, then we convert any direct summands of the form $\mathbb{Z}_{p^{m+1}}$ to the form $\mathbb{Z}_{p^{k+1}}$. If we believe $n \in S_1 - S_2$, we make the number of direct summands of the form $\mathbb{Z}_{p^{m+1}}$ match that in $\mathcal{C}$. If at stage $s-1$, we had none, then we create new ones. If at stage $s-1$, we had too many, then we retain those from the greatest stage $t < s$ where we had the right number (or too few), and convert the extra ones to $\mathbb{Z}_{p^{k+1}}$. If we believe $n \in S_1 \cap S_2$, we make the number of direct summands of the form $\mathbb{Z}_{p^{m+1}}$ match that in $\mathcal{C}^+$. In any case, we add a new direct summand of the form $\mathbb{Z}_{p^{k+1}}$.

□

We turn to the proof of Proposition 5.11. Let $\mathcal{C} = \mathcal{A}_{\omega M}$. By Lemma 5.12, $\mathcal{C}$ has a computable d-Σ_2 Scott sentence. From this, we get a computable d-Σ_{2M+2} sentence describing the groups $\mathcal{G}$ such that $G_{\omega M} \cong \mathcal{C}$. We have a computable Π_{2M+1} sentence characterizing Abelian p-groups with Ulm invariants matching those of $\mathcal{A}$ for $\alpha < \omega M$. The conjunction, which is equivalent to a d-Σ_{2M+2} sentence, is a Scott sentence for $\mathcal{A}$.

For completeness, note that if S is d-Σ^0_{2M+2}, then S is d-Σ^0_2 relative to Δ^0_{2M+1}. Applying Lemma 5.12, we get a uniformly Δ^0_{2M+1} sequence $(\mathcal{C}_n)_{n\in\omega}$ of groups of length N such that $\mathcal{C}_n \cong \mathcal{C}$ iff $n \in S$.

Now, we apply Theorem 5.5, with $\mathcal{C}_n$, together with the Σ^0_{2k+2} relations $R^k_{\mathcal{A}}$, and the Δ^0_{2k+1} functions $f^k_{\mathcal{A}}$, for $k < M$. We get a uniformly computable sequence of groups $(\mathcal{A}_n)_{n\in\omega}$, all of length $\omega M + N$, such that $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$.

□

Proposition 5.13. *Let K be the class of reduced Abelian p-groups of length $\omega M + N$ for some $M, N \in \omega$. Let $\mathcal{A}$ be a computable member of K. If there exist $m < k < N$ such that*

$$u_{\omega M+m}(\mathcal{A}) = u_{\omega M+k}(\mathcal{A}) = \infty \ ,$$

then $I(\mathcal{A})$ is m-complete Π^0_{2M+3} within K.

Proof. We use the following lemma.

Lemma 5.14. *Let $\mathcal{C}$ be a reduced Abelian p-group of length N. Suppose k is greatest such that $u_k(\mathcal{C}) = \infty$, and there exists $m < k$ such that $u_m(\mathcal{C}) = \infty$.*

1. *The structure $\mathcal{C}$ has a computable Π_3 Scott sentence.*

2. *For any X, if S is $\Pi^0_3(X)$, then there is a uniformly X-computable sequence $(\mathcal{C}_n)_{n\in\omega}$, consisting of groups of length N, such that*

$$\mathcal{C}_n \cong \mathcal{C} \textit{ iff } n \in S\ .$$

Proof. For 1, we have a finitary Π_2 sentence describing the reduced Abelian p-groups of length $\leq N$. For each $i < N$, if $u_i(\mathcal{C})$ is finite, we have a finitary d-Σ_2 sentence specifying the value. If $u_i(\mathcal{C}) = \infty$, we have a computable Π_3 sentence saying this. The conjunction is equivalent to a computable Π_3 sentence, and it is a Scott sentence for $\mathcal{C}$.

For 2, assume without loss of generality that $X = \emptyset$. Let S be $\omega - Cof$. We produce a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ of groups of length N such that if $n \in S$, then $\mathcal{C}_n \cong \mathcal{C}$, and if $n \notin S$, say $\omega - W_n$ has cardinality r, then $u_m(\mathcal{C}_n) = r$. We have a sequence of computable approximations to $\omega - W_n$. Let $P_0 = \emptyset$. Given P_s, if there is some $x \in W_{n,s+1} - P_s$, then for the first such x, we let P_{s+1} consist of all $y < x$ in P_s. If there is no such x, then take the first $y \notin W_{n,s+1}$ such that $y \notin P_s$, and let P_{s+1} be the result of adding y to P_s. We have $x \in \omega - W_n$ iff for all sufficiently large s, $x \in P_s$. Moreover, if $\omega - W_n$ is finite, then infinitely often $P_s = W_n$.

We may suppose that

$$\mathcal{C} = H \oplus \mathbb{Z}^{\infty}_{p^{m+1}} \oplus \mathbb{Z}^{\infty}_{p^{k+1}}\ .$$

We construct $\mathcal{C}_n$ as follows. We start with a copy of H. At stage s, say P_s has cardinality r, where at stage $s-1$ the cardinality was r'. If $r' < r$, we add direct summands of the form $\mathbb{Z}_{p^{m+1}}$ to bring the number up to r. If $r' > r$, we keep the direct summands of the form $\mathbb{Z}_{p^{m+1}}$ that we had at the greatest stage $t < s$, where the number was at most r, and we give the remaining ones the form $\mathbb{Z}_{p^{k+1}}$. In any case, we add at least one new direct summand of the form $\mathbb{Z}_{p^{k+1}}$.

□

Now, we turn to the proof of Proposition 5.13. Let $\mathcal{C} = \mathcal{A}_{\omega M}$. By Lemma 5.14, $\mathcal{C}$ has a computable Π_3 Scott sentence. It follows that there is a computable Π_{2M+3} sentence characterizing the groups $\mathcal{G}$ such that $\mathcal{G}_{\omega M} \cong \mathcal{C}$. We have a computable Π_{2M+1} sentence characterizing the Abelian p-groups $\mathcal{G}$ such that for $\alpha < \omega M$, $u_\alpha(\mathcal{G}) = u_\alpha(\mathcal{A})$. There is a computable Π_{2M+3} sentence equivalent to the conjunction, and this is a Scott sentence for $\mathcal{A}$.

For completeness, let S be Π^0_{2M+3}. Then S is Π^0_3 over Δ^0_{2M+1}. By Lemma 5.14, we have a uniformly Δ^0_{2M+1} sequence $(\mathcal{C}_n)_{n\in\omega}$ of groups of length N such that $\mathcal{C}_n \cong \mathcal{C}$ iff $n \in S$. Since $\mathcal{A}$ is computable, we have relations $R^k_{\mathcal{A}}$ and functions $f^k_{\mathcal{A}}$, for $k < M$, as required in Theorem 5.5. We get a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of groups of length $\omega M + N$, such that $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$.

□

We can now summarize the results for groups $\mathcal{A}$ where $\lambda(\mathcal{A}) < \omega^2$.

Theorem 5.15. *Let K be the class of reduced Abelian p-groups of length $\omega M + N$ for some $M, N \in \omega$. Let $\mathcal{A} \in K$.*

1. *If $\mathcal{A}_{\omega M}$ is minimal for the given length (of the form $\mathbb{Z}_{p^N}$), then $I(\mathcal{A})$ is m-complete Π^0_{2M+1} within K.*
2. *If $\mathcal{A}_{\omega M}$ is finite but not minimal for the given length, then $I(\mathcal{A})$ is m-complete d-Σ^0_{2M+1} within K.*
3. *If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{A}) = \infty$, and for all $m < k$, $u_{\omega M+m}(\mathcal{A}) = 0$, then $I(\mathcal{A})$ is m-complete Π^0_{2M+2} within K.*
4. *If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{A}) = \infty$ and for some $m < k$ we have $0 < u_{\omega M+m}(\mathcal{A}) < \infty$, then $I(\mathcal{A})$ is m-complete d-Σ^0_{2M+2} within K.*
5. *If there exist $m < k < N$ such that $u_{\omega M+m}(\mathcal{A}) = u_{\omega M+k}(\mathcal{A}) = \infty$, then $I(\mathcal{A})$ is m-complete Π^0_{2M+3} within K.*

5.3 Groups of Greater Ulm Length

Theorem 5.15 leaves open the possibility, counterintuitive though it may be, that there is an Abelian p-group of length at least ω^2 with an arithmetical index set. The following result rules out this possibility.

Theorem 5.16. *Let $\mathcal{A}$ be a computable reduced Abelian p-group of length greater than ωM. Then for any Δ^0_{2M+1} set S, there is a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that*

$$\mathcal{A}_n \cong \mathcal{A} \text{ iff } n \in S\ .$$

That is, $I(\mathcal{A})$ is Δ^0_{2M+1}-hard.

Proof. Let $\mathcal{C} = \mathcal{A}_{\omega M}$, and let $\mathcal{C}'$ be a finite reduced Abelian p-group, not isomorphic to $\mathcal{C}$. Let S be Δ^0_{2M+1}. We have a uniformly Δ^0_{2M+1} sequence $(\mathcal{C}_n)_{n\in\omega}$ such that $\mathcal{C}_n \cong \mathcal{C}$ if $n \in S$, and $\mathcal{C}_n \cong \mathcal{C}'$ otherwise. Since $\mathcal{A}$ is computable, we have relations $R^k_{\mathcal{A}}$ and functions $f^k_{\mathcal{A}}$ as in Theorem 5.5. Then we get a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$.

□

The following corollary is immediate.

Corollary 5.17. *Let $\mathcal{A}$ be an Abelian p-group of length at least ω^2. Then $I(\mathcal{A})$ is not arithmetical.*

6 Models of the original Ehrenfeucht theory

An Ehrenfeucht theory, is a complete theory T having exactly n non-isomorphic countable models for some finite $n > 1$. A well-known result of Vaught shows that n cannot equal 2 [19]. Ehrenfeucht gave an example for $n = 3$. Ehrenfeucht told Vaught about his example, and it is described in [19]. The language of the theory has a binary relation symbol $<$ and constants c_n for $n \in \omega$. The axioms say that $<$ is a dense linear ordering without endpoints, and the constants are strictly increasing. The theory T has the following three non-isomorphic countable models. There is the prime model, in which there is no upper bound for the constants. There is the saturated model, in which the constants have an upper bound but no least upper bound. There is the *middle* model, in which there is a least upper bound for the constants.

Proposition 6.1. *Let K be the class of models of the original Ehrenfeucht theory T. Let $\mathcal{A}^1$ be the prime model, let $\mathcal{A}^2$ be the middle model, and $\mathcal{A}^3$ be the saturated model.*

1. *$I(\mathcal{A}^1)$ is m-complete Π^0_2 within K.*
2. *$I(\mathcal{A}^2)$ is m-complete Σ^0_3 within K.*
3. *$I(\mathcal{A}^3)$ is m-complete Π^0_3 within K.*

Proof. For 1, first note that there is a computable Π_2 sentence characterizing the models of T such that

$$(\forall x) \bigvee\!\!\!\bigvee_{n\in\omega} x < c_n \ .$$

This is a Scott sentence for $\mathcal{A}^1$. Therefore, $I(\mathcal{A}^1)$ is Π^0_2.

Toward completeness, let S be a Π^0_2 set. We will build a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that

$$\mathcal{A}_n \cong \begin{cases} \mathcal{A}^1 & \text{if } n \in S \ , \\ \mathcal{A}^2 & \text{otherwise} \ . \end{cases}$$

We have a computable approximation $(S_s)_{s\in\omega}$ for S such that

$$n \in S \text{ iff } n \in S_s \text{ for infinitely many } s \ .$$

For fixed n, when $n \notin S$, we build the middle model by creating a least upper bound for the constants we have placed so far and preserving it until/unless our approximation changes. When $n \in S_s$, we destroy the current least upper bound and place the next constant at the end of the ordering. If n really is in S, then the sequence of constants is cofinal, and we get a copy of the prime model. If n is not in S, then for some stage s_0, for all $s \geq s_0$, we have $n \notin S_s$, and we will preserve the least upper bound created at stage s_0, so we get a copy of the middle model.

We turn to 2. First, note that there is a computable Σ_3 Scott sentence for $\mathcal{A}^2$, describing a model of T such that

$$(\exists x)\,[\bigwedge\!\!\!\bigwedge_{n\in\omega} x > c_n \;\&\; (\forall y)\,[(\bigwedge\!\!\!\bigwedge_{n\in\omega} y > c_n) \rightarrow y \geq x]].$$

It follows that $I(\mathcal{A}^2)$ is Σ^0_3.

Toward completeness, let S be a Σ^0_3 set. We build a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that

$$\mathcal{A}_n \cong \begin{cases} \mathcal{A}^2 & \text{if } n \in S \ , \\ \mathcal{A}^3 & \text{otherwise} \ . \end{cases}$$

Note that S is Σ^0_2 over Δ^0_2. We have a Δ^0_2 approximation $(S_k)_{k\in\omega}$, such that $n \in S$ iff for all sufficiently large k, $n \in S_k$. Fix n. Then there is a Δ^0_2 sequence of instructions $(i_k)_{k\in\omega}$. We start with an upper bound for the constants. If $n \notin S_k$, then i_k says to destroy the current least upper bound for the constants, moving left, closer to the constants. If $n \in S_k$, then i_k says to preserve the current least upper bound for the constants.

Now, we build the computable model $\mathcal{A}_n$ based on approximations of the sequence of instructions. There are mistakes of two kinds. We may wrongly guess that i_k said to preserve the current least upper bound for the constants. The result is a delay. We may wrongly guess that i_k said to destroy the current least upper bound for the constants. Having introduced a new upper bound to the left of this one, we correct our mistake by putting the next constant to the right of any added elements, so as to preserve the upper bound as in the instruction.

Again, if n is in S, Δ^0_2 will eventually think so, and we will eventually preserve a particular least upper bound for the constants, building the middle model. Otherwise, infinitely often we will create a new upper bound for the constants, moving to the left, closer to the constants. The result is the saturated model.

Finally, we turn to 3. We have a Π^0_3 Scott sentence for $\mathcal{A}^3$, describing a model of T such that

$$(\exists x)\,[\bigwedge\!\!\!\bigwedge_{n\in\omega} x > c_n] \;\&\; (\forall y)\,[\bigwedge\!\!\!\bigwedge_{n\in\omega} y > c_n \Longrightarrow \exists z[\bigwedge\!\!\!\bigwedge_{n\in\omega} z > c_n \& z < y]] \ .$$

It follows that $I(\mathcal{A}^3)$ is Π^0_3.

For completeness, we notice that the proof is the symmetric case of the proof in part 2.

□

References

[1] E. Barker, "Back and forth relations for reduced Abelian p-groups," *Annals of Pure and Applied Logic* 75 (1995), pp. 223–249.

[2] W. Calvert, "The isomorphism problem for classes of computable fields," *Archive for Mathematical Logic* 75 (2004), pp. 327–336.

[3] W. Calvert, "The isomorphism problem for computable Abelian p-groups of bounded length, " *Journal of Symbolic Logic* 70 (2005), pp. 331–345.

[4] W. Calvert, D. Cenzer, V. Harizanov, and A. Morozov, "Δ_2^0 categoricity of Abelian p-groups," preprint.

[5] W. Calvert, D. Cummins, J. F. Knight, and S. Miller, "Comparing classes of finite structures," *Algebra and Logic*, vol. 43(2004), pp. 374–392.

[6] B. F. Csima, A. Montalbán, and R. A. Shore, "Boolean algebras, Tarski invariants, and index sets," to appear in the *Notre Dame Journal of Formal Logic*.

[7] V. P. Dobritsa, "Complexity of the index set of a constructive model," *Algebra and Logic* 22 (1983), pp. 269–276.

[8] S. S. Goncharov and J. F. Knight, "Computable structure and non-structure theorems," *Algebra and Logic* 41 (2002), pp. 351–373 (English translation).

[9] W. Hodges, *A Shorter Model Theory*, Cambridge University Press, 1997.

[10] I. Kaplansky, *Infinite Abelian Groups*, University of Michigan Press, Ann Arbor, 1954.

[11] H. J. Keisler, *Model Theory for Infinitary Logic*, North-Holland, Amsterdam, 1971.

[12] N. G. Khisamiev, "Constructive Abelian groups," *Handbook of Recursive Mathematics* (Yu. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel, editors), vol. 2, North-Holland, Amsterdam, 1998, pp. 1177–1231.

[13] N. G. Khisamiev, "Constructive Abelian p-groups "*Siberian Advances in Mathematics*, vol. 2(1992), pp. 68–113.

[14] S. Lempp and T. Slaman, "The complexity of the index sets of $\aleph_0$-categorical theories and of Ehrenfeucht theories," to appear in the *Advances in Logic* (Proceedings of the North Texas Logic Conference, October 8–10, 2004), Contemporary Mathematics, American Mathematical Society.

[15] A. W. Miller, "On the Borel classification of the isomorphism class of a countable model", *Notre Dame Journal of Formal Logic*, vol. 24(1983), pp. 22–34.

[16] D. E. Miller, "The invariant Π_α^0 separation principle", *Transactions of the American Mathematical Society*, vol. 242(1978), pp. 185–204.

[17] M. Nadel, "Scott sentences and admissible sets," *Annals of Mathematical Logic* 7 (1974), pp. 267–294.

[18] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin, 1987.

[19] R. L. Vaught, "Denumerable models of complete theories," *Infinitistic Methods: Proceedings of the Symposium on Foundations of Mathematics, Warsaw, 1959*, Pergamon Press, 1961, pp. 303–231.

[20] W. White, "On the complexity of categoricity in computable structures," *Mathematical Logic Quarterly* vol. 49 (2003), pp. 603–614.

[21] W. White, *Characterizations for Computable Structures*, PhD dissertation, Cornell University, 2000.