ALGEBRAIC STRUCTURE AND COMPUTABLE STRUCTURE

A Dissertation

Submitted to the Graduate School

of the University of Notre Dame

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

by

Wesley Crain Calvert, B.A., M.S.

Julia F. Knight, Director

Graduate Program in Mathematics

Notre Dame, Indiana

March 2005

ALGEBRAIC STRUCTURE AND COMPUTABLE STRUCTURE

Abstract

by

Wesley Crain Calvert

The problem often arises, given some class of objects, to classify its members up to isomorphism. The goal of classification theory is to determine whether there is a satisfactory classification, and, if so, to give it. For instance, vector spaces over a fixed field are classified by the dimension. We will consider computable structures, i.e. structures with computable atomic diagram.

We write $\mathcal{A}_e$ for the computable structure with index e. If K is a class of structures, we write

$$I(K) = \{e | \mathcal{A}_e \in K\}$$

and define the *isomorphism problem for K* to be

$$E(K) = \{(a, b) | a, b \in I(K) \text{ and } \mathcal{A}_a \simeq \mathcal{A}_b\}.$$

For some classes (graphs, linear orders, Abelian p-groups, etc.), it is known that the isomorphism problem is m-complete Σ^1_1. This thesis describes the addition of new items to this list, and gives the precise complexity for many simpler classes.

Theorem.

1. *If K is the set of computable members of either of the following, then $E(K)$ is m-complete Σ^1_1:*

(a) *Fields of any fixed characteristic*

(b) *Real closed ordered fields*

2. *If K is the set of computable members of either of the following, then $E(K)$ is m-complete Π^0_3:*

   (a) *The class of models of a first-order strongly minimal theory which is not $\aleph_0$-categorical but which has effective elimination of quantifiers and a computable model.*

   (b) *Archimedean real closed ordered fields*

3. *Let α be a computable limit ordinal, and let $\hat{\alpha} = \sup\limits_{\omega\cdot\gamma<\alpha}(2\gamma+3)$. If K_α is the class of reduced Abelian p-groups of length at most α then $E(K_\alpha)$ is $\Pi^0_{\hat{\alpha}}$ complete.*

4. *The isomorphism problem for computable torsion-free Abelian groups is not hyperarithmetical.*

We will also describe related collaborative work in which the author is involved. This includes a different notion of classification, as well as preliminary results on the following: the complexity of the index sets of particular structures, a calculation of the complexity of the set of indices for structures with Scott rank α, where α is either ω_1^{CK} or $\omega_1^{CK}+1$, and the construction of a structure of Scott rank ω_1^{CK} with a strong approximability property.

A. M. G. D.

CONTENTS

SYMBOLS

φ_e	The eth partial recursive function
W_e	The eth computably enumerable set
$\mathcal{A}_a$	The structure whose diagram has characteristic function φ_a
$I(K)$	$\{a \mid \mathcal{A}_a \in K\}$
$E(K)$	$\{(a,b) \mid a,b \in I(K) \ \& \ \mathcal{A}_a \simeq \mathcal{A}_b\}$

ACKNOWLEDGMENTS

Anyone who has ever undertaken a project of this kind understands immediately the depth of my debt to my thesis advisor, Julia F. Knight. From the day I visited her as a prospective student to the present she has been generous with her time, her ideas, and her experience. It is my great honor to thank her here.

I also thank Valentina Harizanov, Andrei Morozov, and Steven Buechler for serving on my committee. All three have been extremely helpful in the preparation of this thesis. Moreover, I have had very interesting conversations with each of them in which I have learned much of the mathematics that I know.

My other collaborators have also been a great help to me. I have enjoyed working with them, I have learned a lot from them, and I thank them. They are Douglas Cenzer, Desmond Cummins, Sergey Goncharov, Oleg Kudinov, Steffen Lempp, Sara Miller, Vadim Puzarenko, Alexandra Shlapentokh, and Jessica M. Young.

Others have also been helpful with particularly extended or frequent conversations, and with especially helpful ideas. In this vein, I thank Howard Becker, Alexander Berenstein, Peter Cholak, William Dwyer, Jacob Heidenreich, Keith Hubbard, George McNinch, Reed Solomon, Sergei Starchenko, Lou van den Dries, and Rebecca Weber.

The total list of people to whom I am indebted would run to an unmanageable length. However, I would still like to thank the following people for helpful conversations and ideas: John Baldwin, Karen Chandler, Harvey Friedman, Su Gao, Noam Greenberg, Denis Hirschfeldt, Greg Hjorth, Carl Jockusch, David Lippel, Kenneth Manders, Juan Migliore, Joseph Mileti, Benjamin Miller, Joseph Miller, Gerald

Sacks, and Richard Shore. I also gladly express here my gratitude to those who taught me these arts in earlier years: Dwayne Fields, Brenda Bratcher, Robert Johnson, Douglas Nelson, and Thomas Bengtson in mathematics; Cleo Harris, Phillip Janes, and Sherri Hunter in writing; and William Yau in scientific research.

During my graduate studies, I have received financial support from NSF grants DMS 9970452, DMS 0139626, DMS 0353748, and NSF bi-national grant DMS 0075899. I gratefully acknowledge the beneficial effect of this support on my training and research, including the present thesis.

Last, but certainly not least, I thank my family and my friends not already mentioned. In particular, but by no means exclusively, the friendship of Julia Bergner, Erica Olson, and Brian Hall has made graduate school not only bearable but often enjoyable. I don't know how many times I have been saved from disaster or given cause for happiness due to the prayers of my parents and grandparents. The love and friendship of Brandy Barnes has often refreshed me in these past few years, and I thank her for these, as well as for consenting to be my wife.

CHAPTER 1

INTRODUCTION

Computable structure theory, a branch of mathematical logic, seeks to describe relationships between algebraic structure and algorithmic properties. We say that a set S is *computabe* if there is some algorithm which will decide, in a finite amount of time, for a given n, whether $n \in S$. A structure (e.g. group, vector space) is said to be *computable* if the set of basic arithmetical facts (technically called the *atomic diagram*; in a vector space, it is the set of true linear equations and inequations) is computable.

Working with computable structures gives a uniform baseline for using algorithmic properties to "measure" algebraic information. That is, we often find that rich algebraic structure is associated with highly undecidable problems. A book describing computable structure theory from this viewpoint is [5].

In the present chapter, some broad motivation will be given for the work in this thesis, and some notation will be fixed. In Chapters 2 and 3, particular application of these concepts will be made to various classes of Abelian groups and fields. Chapter 4 will analyze finite structures from the perspective of computable structure theory, and develop a general method of comparison of classes of structures with broader application. Chapter 5 will describe work on structures of high Scott rank, a subject that bears importantly on questions of classification.

1.1 Basic Techniques for Computable Structures

We can identify a computable structure with a natural number in the following way: there is an algorithm witnessing that the structure is computable, and each algorithm is associated with a natural number, so we identify the structure with the number of its algorithm, writing $\mathcal{A}_e$ for the structure with $\chi_{D(\mathcal{A})} = \varphi_e$. Of course, it is possible that $e_1 \neq e_2$ but $\mathcal{A}_{e_1} \simeq \mathcal{A}_{e_2}$ (or even $\mathcal{A}_{e_1} = \mathcal{A}_{e_2}$).

We will also make extensive use of computable infinitary formulas. Since nothing will depend on the choice of ordinal notations, we will in all cases assume that we are working below some (large) computable ordinal and fix some particular set of notations for ordinals below it.

Definition 1.1.1. *We define the set of computable Σ_α and Π_α formulas inductively on α.*

1. *The Σ_0 and Π_0 formulas coincide, and are the finitary quantifier-free formulas.*
2. *For $\alpha > 0$, a computable Σ_α formula is a disjunction of a c.e. set of formulas φ_i, where φ_i is of the form $\exists x \psi_i$, where ψ_i is Π_{β_i} for some $\beta_i < \alpha$, and where for any $\gamma < \alpha$ there is some i such that $\beta_i \geq \gamma$.*
3. *For $\alpha > 0$, a computable Π_α formula is a conjunction of a c.e. set of formulas φ_i, where φ_i is of the form $\forall x \psi_i$, where ψ_i is Σ_{β_i} for some $\beta_i < \alpha$, and where for any $\gamma < \alpha$ there is some i such that $\beta_i \geq \gamma$.*

1.2 Classification of Computable Structures

In many areas of mathematics the problem arises, given some class of objects, to classify them up to isomorphism. In some cases, we expect that there is some reasonable solution, while in others there seems unlikely to be any satisfactory classification. The goal of classification theory is to determine whether there is a satisfactory classification, and, if so, to give it. For instance, vector spaces over a fixed field are classified up to isomorphism by a single cardinal, the dimension.

Approaches to classification in model theory [66] and set theory [25, 38] have given important progress on these issues. Approaches which focus on computable structures sometimes reveal finer structure. Also, they can show special consequences of the hypothesis that our models be computable.

One way to think about classification of computable structures is through index sets. We can identify a class of computable structures with the set

$$I(K) = \{e | \mathcal{A}_e \in K\}$$

and the equivalence relation of isomorphism becomes the set

$$E(K) = \{(a, b) | a, b \in I(K) \text{ and } \mathcal{A}_a \simeq \mathcal{A}_b\}$$

called the *isomorphism problem for* K.

If $I(K)$ is hyperarithmetical, which, in practice, is usually a mild assumption, then $E(K)$ is, at worst, Σ^1_1. That is, it may be described by some statement involving a single existential quantifier ranging over functions and perhaps some quantifiers ranging over numbers. For instance, $E(K)$ may be described by stating that there is a function from $\mathcal{A}_a$ to $\mathcal{A}_b$ which is bijective and respects all of the structure. If $E(K)$ has a simpler description — perhaps one involving only number quantifiers — we take this to be a classification. Goncharov and Knight showed that this notion is closely related to other promising notions of classification for computable structures [31]. We may focus on a class of structures, differing in isomorphism type, or we may focus on a single isomorphism type.

For some classes (graphs, linear orders, Boolean algebras, Abelian p-groups, etc.) that are non-classifiable from other points of view, it is known from folklore that the isomorphism problem is m-complete Σ^1_1. In Chapter 3 of the present work, more items will be added to this list, and at various points the m-degree of the isomorphism problem will be computed for other classes of structures.

One further issue will be relevant: context. Sometimes describing that the structure in question is, for instance, a vector space, is sufficiently difficult to swamp the complexity of a description that distinguishes it from other vector spaces. It is this second description that interests us, since in practice one typically knows what type of structure one will encounter (a vector space, a graph, a group, etc.).

Definition 1.2.1. *Suppose $A \subseteq B$. Let Γ be some complexity class (e.g. Π^0_3), and let $S \subseteq \omega$. We define complexity* within B*:*

1. *A is Γ* within B *if and only if there is some $R \in \Gamma$ such that $A = R \cap B$.*
2. *$S \leq_m A$* within B *if there is a computable $f : \omega \to B$ such that for all n, $n \in S \iff f(n) \in A$.*
3. *A is m-*complete Γ within B *if A is Γ within B and for any $S \in \Gamma$ we have $S \leq_m A$ within B.*

The last part of the definition says that A is Γ complete within B if it is Γ within B and there is a function witnessing that it is Γ complete which only calls for questions about things in B. In fact, the questions are only about members of a c.e. subset of B. We will sometimes write "within K" for "within $I(K)$" or "within $I(K) \times I(K)$."

CHAPTER 2

ALGEBRAIC AND ALGORITHMIC PROPERTIES OF ABELIAN GROUPS

Abelian groups are well-travelled terrain in logic. They seem to have a good mix of being sufficiently determined by well-understood information (roughly, order and divisibility of elements) and yet admitting enough variation to be interesting. The case of torsion groups, particularly Abelian p-groups, is especially noteworthy in this respect.

In this chapter, we will calculate the degree of the isomorphism problem for various classes of Abelian p-groups and the degrees of the index sets of many such groups. Finally, we will see the little that is known about the isomorphism problem for torsion-free Abelian groups.

2.1 Notation and Terminology for Abelian p-Groups

Let p be an arbitrary prime number. Abelian p-groups are Abelian groups in which each element has some power of p for its order. We will consider only countable Abelian p-groups. These groups are of particular interest because of their classification up to isomorphism by Ulm. For a classical discussion of this theorem and a more detailed discussion of this class of groups, consult Kaplansky's book [39]. Generally, notation here will be similar to Kaplansky's.

It is often helpful to follow S. Feferman [22] in representing these groups by trees. Consider a tree T. The Abelian p-group $G(T)$ is the group generated by the nodes in T (among which the root is 0), subject to the relations stating that the group is Abelian and that px is the predecessor of x in the tree. Reduced Abelian p-groups, from this perspective, are represented by trees with no infinite paths.

The idea of Ulm's theorem is that it generalizes the notion that to determine a finitely generated torsion Abelian group it is only necessary to determine how many cyclic components of each order are included in a direct sum decomposition. Let G be a countable Abelian p-group. We will produce an ordinal sequence (usually transfinite) of cardinals $u_\beta(G)$ (each at most countable), which is constant after some ordinal (called the "length" of G). If H is also a countable Abelian p-group and for all β we have $u_\beta(G) = u_\beta(H)$, then $H \simeq G$ (this is still subject to another condition we have yet to define).

First, set $G_0 = G$. Now we inductively define $G_{\beta+1} = pG_\beta = \{px | x \in G_\beta\}$, where px denotes the sum of x with itself p times. We also define, for limit β, the subgroup $G_\beta = \bigcap_{\gamma<\beta} G_\gamma$. Further, let $P(G)$ denote the subgroup of elements x for which $px = 0$, and let $P_\beta(G) = P \cap G_\beta$. Now the quotient $P_\beta(G)/P_{\beta+1}(G)$ is a $\mathbb{Z}_p$ vector space, and we call its dimension $u_\beta(G)$. Where no confusion is likely, we will omit the argument G and simply write P_β, and so forth.

For any countable Abelian p-group G, there will be some least ordinal $\lambda(G)$ such that $G_{\lambda(G)} = G_{\lambda(G)+1}$. This is the *length* of G. If $G_{\lambda(G)} = \{0\}$, then we say that G is *reduced*. Equivalently, G is reduced if and only if it has no divisible subgroup. The *height* of an element x is the unique β such that $x \in G_\beta$, but $x \notin G_{\beta+1}$, provided that such a β exists. It is conventional to write $h(0) = \infty$, where ∞ is greater than any ordinal. Similarly, if our group contains a divisible element x, we write $h(x) = \infty$. In the course of this paper, we will only consider reduced groups. When G is a direct sum of cyclic groups, $u_n(G)$ is exactly equal to the number of direct summands of order p^{n+1}. We can now state Ulm's theorem, but we will not prove it here.

Theorem 2.1.1 (Ulm). *Let G and H be countable reduced countable Abelian p-groups. Then $G \simeq H$ if and only if for every countable ordinal β we have $u_\beta(G) = u_\beta(H)$.*

It is interesting to note that this theorem is not "recursively true." Lin showed that if two computable groups satisfying the hypotheses of this theorem have identical Ulm invariants, they may not be computably isomorphic [46]. In reverse mathematics, there is a theorem stating that (depending heavily on the particular statement of the theorem), Ulm's theorem is equivalent over a weak base system to a formal system called ATR_0, corresponding roughly to the existence of all countable ordinals [24, 67]. Related work from a constructivist perspective may be found in a paper by Richman [62].

A calculation of the complexity of the isomorphism problem for special classes of computable reduced Abelian p-groups is essentially a computation of the complexity of checking the equality of Ulm invariants. Given some computable ordinal α, we will consider the class of reduced Abelian p-groups of length at most α.

2.2 The Isomorphism Problem for Ableian p-Groups of Bounded Ulm Length

In view of Ulm's theorem, and in view of previous work by Barker [9], Nadel [56], and others on the expressibility of Ulm information in $L_{\omega_1\omega}$, it seems reasonable to try to calculate the complexity of the isomorphism problem for Abelian p-groups. Getting upper bounds on the complexity is not difficult. The completeness results for lengths less than ω^2 will follow from a theorem of N. Khisamiev, which characterizes which Abelian p-groups of small Ulm length have computable copies. For greater lengths, completeness proofs will use the Ash Metatheorem. The results of this section will appear in [13].

2.2.1 Bounds on Isomorphism Problems

For any computable ordinal α, it is somewhat straightforward to write a computable infinitary sentence stating that G is a reduced Abelian p-group of length at most α and that G and H have the same Ulm invariants up to α. In particular, Barker [9] verified the following.

Lemma 2.2.1. *Let G be a computable Abelian p-group.*

1. $G_{\omega\cdot\alpha}$ *is* $\Pi^0_{2\alpha}$.
2. $G_{\omega\cdot\alpha+m}$ *is* $\Sigma^0_{2\alpha+1}$.
3. $P_{\omega\cdot\alpha}$ *is* $\Pi^0_{2\alpha}$.
4. $P_{\omega\cdot\alpha+m}$ *is* $\Sigma^0_{2\alpha+1}$.

Proof. It is easy to see that 3 and 4 follow from 1 and 2 respectively. Toward 1 and 2, note the following:

$$\begin{aligned}
x \in G_m &\iff \exists y(p^m y = x)\\
x \in G_\omega &\iff \bigwedge\!\!\!\bigwedge_{m\in\omega} \exists y(p^m y = x)\\
x \in G_{\omega\cdot\alpha+m} &\iff \exists y[p^m y = x \wedge G_{\omega\cdot\alpha}(y)]\\
x \in G_{\omega\cdot\alpha+\omega} &\iff \bigwedge\!\!\!\bigwedge_{m\in\omega} \exists y[p^m y = x \wedge G_{\omega\cdot\alpha}(y)]\\
x \in G_{\omega\cdot\alpha} &\iff \bigwedge\!\!\!\bigwedge_{\gamma<\alpha} G_{\omega\cdot\gamma}(x) \text{ for limit } \alpha
\end{aligned}$$

$\square$

Work by Lin [47], when viewed from our perspective, shows that for any $m \in \omega$, there is a group G in which G_m is Σ^0_1 complete. Given this lemma, we can place bounds on the complexity of $I(K)$ and $E(K)$.

Lemma 2.2.2. *If K_α is the class of reduced Abelian p-groups of length at most α, and $\beta > 0$ is a computable ordinal, then $I(K_{\omega\cdot\beta+m})$ is $\Pi^0_{2\beta+1}$.*

Proof. The class $K_{\omega\cdot m+\beta}$ may be characterized by the axioms of Abelian p-groups (which are Π^0_2), together with an axiom saying

$$\forall x[x \in G_{\omega\cdot\beta+m} \rightarrow x = 0]$$

Since the previous lemma guarantees that this sentence is $\Pi_{2\beta+1}$, we know that $I(K_{\omega\cdot\beta+m})$ is also $\Pi^0_{2\beta+1}$. $\square$

Lemma 2.2.3. *If $\alpha > 0$ is a computable ordinal and K_α is as in the previous lemma, we use $\hat{\alpha}$ to denote $\sup\limits_{\omega\cdot\gamma<\alpha}(2\gamma+3)$. Then $E(K_\alpha)$ is $\Pi^0_{\hat{\alpha}}$ within K_α.*

Proof. Note that the relation "there are at least n elements of height β which are $\mathbb{Z}_p$-independent over $G_{\beta+1}$" is defined in the following way. To say that $x_1, \ldots, x_n$ are $\mathbb{Z}_p$-independent over $G_{\beta+1}$, we write the computable $\Pi_{2\beta+1}$ formula

$$D_{n,\beta}(x_1, \ldots, x_n) = \bigwedge_{\substack{b_1,\ldots b_n \in \mathbb{Z}_p \\ \vec{b}\neq 0}} (\sum_{i=1}^{n} b_i x_i \notin G_{\beta+1})$$

Now to write "there are at least n independent elements of height β and order p," we use the sentence

$$B_{n,\beta} = \exists x_1, \ldots, x_n[(\bigwedge_{i=1}^{n} G_\beta(x_i)) \wedge (\bigwedge_{i=1}^{n} px_i = 0)) \wedge D_{n,\beta}(\overline{x})]$$

which is a computable $\Sigma_{2\beta+2}$ sentence. Now we can define isomorphism by

$$\bigwedge\!\!\!\bigwedge_{\substack{n\in\omega \\ \beta<\alpha}} \mathcal{A}_a \models B_{n,\beta} \Leftrightarrow \mathcal{A}_b \models B_{n,\beta}$$

We write each $\beta < \alpha$ as $\beta = \omega \cdot \gamma + m$, where $m \in \omega$. If $\hat{\alpha}$ is as defined in the statement of the lemma, then this can be expressed by a computable $\Pi_{\hat{\alpha}}$ sentence. $\square$

2.2.2 Completeness for Length $\omega \cdot m$

Proposition 2.2.4. *If K_ω is the class of computable Abelian p-groups of length at most ω, then $E(K_\omega)$ is Π^0_3 complete within K_ω.*

Proof. We first observe that the set is Π^0_3 within K, by applying the previous lemma. Now let S be an arbitrary Π^0_3 set. We can represent S as the set defined by

$$\forall e \exists^{<\infty} y \; R(n, e, y)$$

where R is computable and $\exists^{<\infty}$ is read "there exist at most finitely many." Consider the Abelian p-group G^ω of length ω and with Ulm sequence $u_\alpha = \omega$ for all $\alpha < \omega$.

We will build a uniformly computable sequence $(H^n)_{n\in\omega}$ of reduced Abelian p-groups of height at most ω such that $H^n \simeq G^\omega$ if and only if $n \in S$. Let $G^{\omega,\infty}$ denote the direct sum of countably many copies of the smallest divisible Abelian p-group $\mathbb{Z}(p^\infty)$, and note that $G^{\omega,\infty}$ has a computable copy, as a direct sum of copies of a subgroup of $\mathbb{Q}/\mathbb{Z}$. We will denote the element where x occurs in the ith place with zeros elsewhere by $(x)_i$. For instance, set-wise, $G^{\omega,\infty}$ is the collection of all sequences of proper fractions whose denominators are powers of p, and the element $(\frac{1}{p})_2$ denotes the element $(0, \frac{1}{p}, 0, 0, \ldots)$.

List the atomic sentences by $(\phi_e)_{e\in\omega}$, the pairs of elements in $G^{\omega,\infty}$ by ξ_e, and set $D_{-1} = C_{-1} = Y_{e,-1} = X_{e,-1} = \tilde{X}_{e,-1} = T_{e,-1} = \emptyset$. We will build groups to meet the following requirements:

- P_e : There are infinitely many independent elements $x \in H^n$ of order p and height exactly e if and only if there are at most finitely many y such that $R(n, e, y)$.
- Q_e : If $\xi_e = (a, b)$ and $a, b \in H^n$, then $a + b \in H^n$.
- Z_e : If all parameters occurring in ϕ_e are in H^n, then exactly one of $\phi_e \in D$ or $\neg\phi_e \in D$.

Roughly speaking, D_s will be the diagram of H^n, and C_s will be its domain. For each e, the set $Y_{e,s}$ will keep track of the y already seen, $X_{e,s}$ the x created of height at least e, and $\tilde{X}_{e,s}$ the x which are given greater height, as in P_e. The set $T_{e,s}$ will keep track of the heights greater than e already used to put elements from X_e in $\tilde{X}_e$, so that we do not accidentally make infinitely many elements of height $e+1$.

We say that P_e requires attention at stage s if there is some $y < s$ such that $y \notin Y_{e,s-1}$ and $R(n,e,y)$ and there is also some $x \in X_{e,s-1} \setminus \tilde{X}_{e,s-1}$, or if for all $y < s$ we have either $y \in Y_{e,s-1}$ or $\neg R(n,e,y)$. We say that Q_e requires attention at stage s if $\xi_e = (a,b)$ and $a, b \in C_{s-1}$ but $a + b \notin C_{s-1}$. We say that Z_e requires attention at stage s if all parameters that occur in ϕ_e are in C_{s-1} and D_{s-1} does not include either ϕ_e or $\neg\phi_e$.

At stage s, to satisfy P_e, we will act by first looking for some $y < s$ such that $y \notin Y_{e,s-1}$ and $R(n,e,y)$. If none is found, the action will be to enumerate a new independent x of height at least e. To do this, find the first k such that $(\frac{1}{p})_k$ does not occur in C_{s-1} or in any element of D_{s-1}. Let

$$C_s = C_{s-1} \cup \left\{ \left(\frac{1}{p^j}\right)_k | j = 1, \ldots, (e-1) \right\}$$

and set $X_{e,s} = X_{e,s-1} \cup \{(\frac{1}{p})_k\}$, $\tilde{X}_{e,s} = \tilde{X}_{e,s-1}$, $T_{e,s} = T_{e,s-1}$, and $Y_{e,s} = Y_{e,s-1}$. If such a y is found, on the other hand, the action will be to give all existing element of $X_{e,s-1}$ height greater than e. To do this, collect

$$K = \left\{ k | \left(\frac{1}{p}\right)_k \in X_{e,s-1} \setminus \tilde{X}_{e,s-1} \right\}$$

and the least positive $r \notin T_{e,s-1}$. Note that K is finite. Set

$$C_s = C_{s-1} \cup \bigcup_{k \in K} \left\{ \left(\frac{1}{p^j}\right)_k | j = (e, \ldots, e+r+1) \right\}$$

and set $T_{e,s} = T_{e,s-1} \cup \{r\}$, $\tilde{X}_{e,s} = \tilde{X}_{e,s-1} \cup \{(\frac{1}{p})_k | k \in K\}$, $X_{e,s} = X_{e,s-1}$, and $Y_{e,s} = Y_{e,s-1} \cup \{y\}$.

To satisfy Q_e at stage s we will look to see whether the elements of $\xi_e = (a, b)$ are in C_{s-1}. If they are both there, set $C_s = C_{s-1} \cup \{a + b\}$. Otherwise, set $C_s = C_{s-1}$.

To satisfy Z_e, we will act at stage s by first looking for the parameters in ϕ_e in C_{s-1}. If all of them are there and $G^{\omega,\infty} \models \phi_e$, then set $D_s = D_{s-1} \cup \{\phi_e\}$. If all of them are there and $G^{\omega,\infty} \models \neg\phi_e$, then set $D_s = D_{s-1} \cup \{\neg\phi_e\}$. If some of the parameters are not in C_{s-1}, we set $D_s = D_{s-1}$.

Now if $n \in S$, for each e we have Q_e to guarantee that $u_e(H^n)$ will be infinite, so $H^n \simeq G^\omega$. If $n \notin S$, there is some e such that Q_e guarantees that $u_e(H^n)$ is finite, so $H^n \not\simeq G^\omega$. □

Since this result is perfectly uniform, we can use it for induction. What we actually have established is the following:

Proposition 2.2.5. *If S is a set which is Π^0_3 relative to X, then there is a uniformly X-computable sequence of reduced Abelian p-groups $(H^n)_{n\in\omega}$, each of length at most ω, such that $H^n \simeq G^\omega$ if and only if $n \in S$.*

There is a result of N. Khisamiev [41], which allows us to transfer these X-computable groups down to the computable level. Although we only use the weaker corollary here, the result is stated in its full strength because it will be needed in a later section.

Theorem 2.2.6 (N. Khisamiev [41]). *For a reduced Abelian p-group $\mathcal{A}$ of height $\omega \cdot M$, where $M \in \omega$, the following are equivalent:*

1. *$\mathcal{A}$ has an X-computable copy.*
2. *For each $k < N$, the relation*

$$R^k_{\mathcal{A}} = \{(r, t) | u_{\omega k + r}(\mathcal{A}) \geq t\}\}$$

is $\Sigma^0_{2k+2}(X)$, and there is a $\Delta^0_{2k+1}(X)$ function $f^k_{\mathcal{A}}(r, s)$ such that for any fixed *r, the function $f^k_{\mathcal{A}}(r,s)$ is nondecreasing and $\lim_s f(r,s) = r^* \geq r$ exists, with $u_{\omega k + r^*}(\mathcal{A}) \neq 0$.*

Moreover, we can pass effectively from $\Sigma^0_{2k+2}(X)$ indices for the relations $R^k_{\mathcal{A}}$ and $\Delta^0_{2k+1}(X)$ indices for the functions $f^k_{\mathcal{A}}$ to an X-computable index for an X-computable copy of $\mathcal{A}$, and conversely.

Corollary 2.2.7 (N. Khisamiev [41]). *If G is a X''-computable reduced Abelian p-group, then there is an X-computable reduced Abelian p-group H such that $H_\omega \simeq G$ and $u_n(H) = \omega$ for all $n \in \omega$. Moreover, from an index for G, we can effectively compute an index for H.*

Of course, the theorem would allow many other choices for $u_n(H)$, beyond what we need here. We can now combine Proposition 2.2.5 and Corollary 2.2.7 to obtain the following completeness result.

Proposition 2.2.8. *If $K_{\omega\cdot m}$ is the class of computable reduced Abelian p-groups of length at most $\omega \cdot m$, for some $m > 0$, then $E(K_{\omega\cdot m})$ is m-complete Π^0_{2m+1} within K.*

Proof. Let S be an arbitrary Π^0_{2m+1} set. Since S is Π^0_3 in $\emptyset^{(2m-1)}$, we have a uniformly $\emptyset^{(2m+1)}$-computable sequence of reduced Abelian p-groups $(H^n)_{n\in\omega}$, each of length at most ω, such that $H^n \simeq G^\omega$ if and only if $n \in S$. Now we can step each H^n down to a lower level using Khisamiev's result, so that we have a uniformly $\emptyset^{(2n-3)} = \emptyset^{(2(n-1)-1)}$-computable sequence $(H^{2,n})_{n\in\omega}$ of reduced Abelian p-groups, each of height $\omega \cdot 2$ which again have the property that $H^{2,n}$ has a constantly infinite Ulm sequence if and only if $n \in S$. By induction, we define $(H^{i,n})_{n\in\omega}$, and when we get to $(H^{m,n})_{n\in\omega}$, it will be a uniformly computable sequence of groups of length at most $\omega \cdot m$ such that $H^{m,n}$ has constantly infinite Ulm sequence if and only if $n \in S$. $\square$

2.2.3 Completeness for Higher Bounds on Length

Giving completeness results for higher levels requires more elaborate machinery. We will prove a more general result using an α-system, in the sense of Ash. These systems are explained in detail, along with several other variants, in the book of Ash and Knight [5]. The "Metatheorem" for α-systems was originally proved in a paper by Ash [2]. The version presented here was stated in Ash's later paper [3].

Roughly speaking, an α-system describes all possible priority constructions of a given kind, and the Metatheorem states that given an "instruction function" which is Δ^0_α, the system will produce a c.e. set (in our case, the diagram of a group) which incorporates the information given in the instruction function. More formally, we make the following definition:

Definition 2.2.9 (Ash). *Let α be a computable ordinal. An α-system is a structure*

$$(L, U, P, \hat{\ell}, E, (\leq_\beta)_{\beta<\alpha})$$

where L and U are c.e. sets, E is a partial computable "enumeration" function on L, P is a c.e. "alternating tree" on L and U (that is, a set of strings closed under nonempty initial segments, and with letters alternating between L and U) in which all members start with $\hat{\ell} \in L$, and $\leq_\beta$ are uniformly c.e. binary relations on L, where the following properties are satisfied:

1. *$\leq_\beta$ is reflexive and transitive for all $\beta < \alpha$*
2. *$a \leq_\gamma b \Rightarrow a \leq_\beta b$ for all $\beta < \gamma < \alpha$*
3. *If $a \leq_0 b$, then $E(a) \subseteq E(b)$*
4. *If $\sigma u \in P$, where σ ends in ℓ^0, and*

$$\ell^0 \leq_{\beta_0} \ell^1 \leq_{\beta_1} \cdots \leq_{\beta_{k-1}} \ell^k$$

where $\beta_0 > \beta_1 > \cdots > \beta_k$, then there exists some ℓ^* such that $\sigma u \ell^* \in P$ and for all $i \leq k$, we have $\ell^i \leq_{\beta_i} \ell^*$.

In the systems we will use here, E will eventually enumerate the diagram of the structure we are building. If we have such a system, we say that an *instruction function* for P is a function q from the set of sequences in P of odd length (i.e. those with a last term in L) to U, so that for any σ in the domain of q, $\sigma q(\sigma) \in P$. The following theorem, due to Ash [3], guarantees that if we have such a function, there is a string which represents "carrying out" the instructions while enumerating a c.e. set. We call an infinite string $\pi = \hat{\ell} u_1 \ell_1 u_2 \ell_2 \dots$ a "run" of (P, q) if it is a path through P with the property that for any initial segment σu we have $u = q(\sigma)$. The Metatheorem also guarantees that there is a run with the property that $\bigcup_{i \in \omega} E(\ell_i)$ is computably enumerable.

Proposition 2.2.10 (Ash Metatheorem). *If we have an α-system*

$$(L, U, P, \hat{\ell}, E, (\leq_\beta)_{\beta < \alpha})$$

and if q is a Δ^0_α instruction function for P, then there is a run $\pi : \omega \to (L \cup U)$ of (P, q) such that $\bigcup_{i \in \omega} E(\pi(2i))$ is c.e. Further, from computable and c.e. indices for the components of the system and a Δ^0_α index for q, we can effectively determine a c.e. index for $\bigcup_{i \in \omega} E(\pi(2i))$.

What this means is that if we can set up an appropriate system, then given some highly undecidable requirements, we can build a computable group to satisfy them. The difficulty (aside from digesting the Metatheorem itself) mainly consists of defining the right system. Afterwards, it is no trouble to write out the high-level requirements we want to meet. Using such a system, we will prove the following generalization of Proposition 2.2.8.

Theorem 2.2.11. *Let $\alpha > 0$ be a computable limit ordinal, and let*

$$\hat{\alpha} = \sup_{\omega\cdot\gamma<\alpha} (2\gamma + 3)$$

as in Proposition 2.2.3. If K_α is the class of reduced Abelian p-groups of length at most α then $E(K_\alpha)$ is $\Pi^0_{\hat{\alpha}}$ complete within K_α.

Proof. Let $(\alpha_i)_{i\in\omega\setminus\{0\}}$ be a computable sequence of computable ordinals, cofinal in α (for instance, if $\alpha = \omega \cdot \omega$, then $\alpha_i = \omega \cdot i$ would do, or if $\alpha = \omega \cdot (\beta + 1)$, we could use $\alpha_i = \omega \cdot \beta + i$; in any case, since α is computable, there is such a sequence). Consider the family of groups $(\hat{G}^i)_{i\in\omega}$, each of length α where $\hat{G}_0$ has uniformly infinite Ulm sequence and

$$u_\beta(\hat{G}^i) = \begin{cases} \omega & \text{if } \beta < \alpha_i \text{ or if } \beta \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Since the Ulm sequences of these groups are uniformly computable, there is a uniformly computable sequence $(G^i)_{i\in\omega}$ such that $G^i \simeq \hat{G}^i$ for all i, and such that in each of these groups, for any β, the predicate "x has height β" is computable. The proof of this, which is due to Oates, is a modification of an argument of L. Rogers [64], and may be found in Barker's paper [9].

For any set $S \in \Pi^0_{\hat{\alpha}}$, we will construct a sequence of groups $(H^n)_{n\in\omega}$ such that if $n \in S$ then $H^n \simeq G^0$, and otherwise, $H^n \simeq G^i$ for some $i \neq 0$. To do this, we will define an $\hat{\alpha}$-system. Let L be the set of pairs (j, p), where $j \in \omega$ and p is a finite injective partial function from ω to G^j. Let U be the set $\{0, 1\}$. Use n_p to denote $|dom(p)|$. By $E(j, p)$, we will mean the first n_p atomic sentences or negations of atomic sentences with parameters from the image of p which are true in G^j. Let $\hat{\ell} = (0, \emptyset)$, and P be the set of strings of the form $\hat{\ell}u_1\ell_1u_2\ell_2\ldots$ which satisfy the following properties:

1. $u_i \in U$ and $\ell_i \in L$

2. If $u_i = 1$ then $u_{i+1} = 1$

3. If $\ell_i = (j_i, p_i)$, then both the domain and range of p_i contain at least the first i members of ω

4. If $\ell_i = (j, p)$ and $u_i = 1$, then $j \neq 0$. Otherwise, $j = 0$. Further, if $u_{i-1} = 1$ and $\ell_{i-1} = (j_{i-1}, q)$, then $j = j_{i-1}$.

For the $\leq_\beta$ we will modify the standard back-and-forth relations on Abelian p-groups. In general, the standard back-and-forth relations on a class K are characterized as relations on pairs $(\mathcal{A}, \overline{a})$ where $\mathcal{A} \in K$ and $\overline{a}$ is a finite tuple of $\mathcal{A}$.

Definition 2.2.12. *If $\overline{a} \subseteq \mathcal{A}$ and $\overline{b} \subseteq \mathcal{B}$ are finite tuples of equal length, then we define the* standard back-and-forth relations $\leq_\beta$ *as follows:*

1. *$(\mathcal{A}, \overline{a}) \leq_1 (\mathcal{B}, \overline{b})$ if and only if for all finitary Σ^0_1 formulas true of $\overline{b}$ in $\mathcal{B}$ are true of $\overline{a}$ in $\mathcal{A}$.*

2. *$(\mathcal{A}, \overline{a}) \leq_\beta (\mathcal{B}, \overline{b})$ if and only if for any finite $\overline{d} \subset \mathcal{B}$ and any γ with $1 \leq \gamma < \beta$ there is some $\overline{c} \subset \mathcal{A}$ of equal length such that $(\mathcal{B}, \overline{b}, \overline{d}) \leq_\gamma (\mathcal{A}, \overline{a}, \overline{c})$.*

This definition extends naturally to tuples of different length as follows: we say that $(\mathcal{A}, \overline{a}) \leq_\beta (\mathcal{B}, \overline{b})$ if and only if $\overline{a}$ is no longer than $\overline{b}$ and that for the initial segment $\overline{b}' \subset \overline{b}$ of length equal to that of $\overline{a}$, we have $(\mathcal{A}, \overline{a}) \leq_\beta (\mathcal{B}, \overline{b}')$. Barker [9] gave a useful characterization of these relations in the case of Abelian p-groups $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A} = \mathcal{B}$.

Proposition 2.2.13 (Barker [9]). *If $\leq_\beta$ are the standard back-and-forth relations on reduced Abelian p-groups, and if $\overline{a}$ and $\overline{b}$ are finite subsets of equal length in an Abelian p-group with the height of elements given by h respectively and with equal cardinality, with a function f mapping elements of $\bar{b}$ to corresponding elements of $\bar{a}$, then the following hold:*

1. *$\overline{a} \leq_{2\cdot\delta} \overline{b}$ if and only if the two generate isomorphic subgroups and for every $b \in \overline{b}$ and $a = f(b)$ we have*

$$h(a) = h(b) < \omega \cdot \delta \text{ or } h(b), h(a) \geq \omega \cdot \delta$$

2. *$\overline{a} \leq_{2\cdot\delta+1} \overline{b}$ if and only if the two generate isomorphic subgroups and for every $b \in \overline{b}$ and $a = f(b)$ we have*

   (a) *In the case that $P_{\omega\cdot\delta+k}$ is infinite for every $k \in \omega$,*

   $$h(a) = h(b) < \omega \cdot \delta$$

   or

   $$h(b) \geq \omega \cdot \delta \text{ and } h(a) \geq \min\{h(b), \omega \cdot \delta + \omega\}$$

   (b) *In the case that $P_{\omega\cdot\delta+k}$ is infinite and $P_{\omega\cdot\delta+k+1}$ is finite,*

   $$h(a) = h(b) < \omega \cdot \delta$$

   or

   $$\omega \cdot \delta \leq h(b) \leq h(a) \leq \omega \cdot \delta + k$$

   or

   $$h(a) = h(b) > \omega \cdot \delta + k$$

   (c) *In the case that $P_{\omega\cdot\delta}$ is finite,*

   $$h(x) = h(x)$$

Since in all groups with which we are concerned, $P_{\omega\cdot\delta+k}$ will be infinite for all $\delta < \alpha$, we will have no need for the more complicated cases. Also, it is helpful to deal with groups which satisfy the stronger condition that they have infinite Ulm invariants at each limit level. For the proof of the present theorem, we do not actually need the standard back and forth relations, but only relations which satisfy the hypotheses of the Ash Metatheorem, including the back and forth property. In exchange, however, we need a system that considers tuples from *different* groups.

Definition 2.2.14. *Let $\mathcal{A}, \mathcal{B}$ be countable reduced Abelian p-groups of length at most α such that for any limit ordinal $\nu < \alpha$ we have $u_\nu(\mathcal{A}) = u_\nu(\mathcal{B}) = \omega$. Let the height of an element in its respective group be given by h. Let $\overline{a}, \overline{b}$ be finite sequences of equal length from $\mathcal{A}$ and $\mathcal{B}$, respectively. Then define $(\leq_\delta)_{\delta<\omega_1}$ by the following:*

1. $(\mathcal{A}, \overline{a}) \leq_{2\cdot\delta} (\mathcal{B}, \overline{b})$ *if and only if*

   (a) *The function matching elements of $\overline{a}$ to corresponding elements of $\overline{b}$ extends to an isomorphism $f : \langle\overline{b}\rangle \to \langle\overline{a}\rangle$,*

   (b) *for every $b \in \overline{b}$ and $a = f(b)$ we have*

   $$h(a) = h(b) < \omega \cdot \delta \text{ or } h(b), h(a) \geq \omega \cdot \delta$$

   and

   (c) *for all $\beta < \omega \cdot \delta$ we have $u_\beta(\mathcal{A}) = u_\beta(\mathcal{B})$.*

2. $(\mathcal{A}, \overline{a}) \leq_{2\cdot\delta+1} (\mathcal{B}, \overline{b})$ *if and only if*

   (a) *The function matching respective elements in $\overline{a}$ and $\overline{b}$ extends to an isomorphism $f : \langle\overline{b}\rangle \to \langle\overline{a}\rangle$,*

   (b) *for every $b \in \overline{b}$ and $a = f(b)$ we have*

   $$h(a) = h(b) < \omega \cdot \delta$$

   or

   $$h(b) \geq \omega \cdot \delta \text{ and } h(a) \geq \min\{h(b), \omega \cdot \delta + \omega\}$$

   (c) *for all $\beta < \omega \cdot \delta$ we have $u_\beta(\mathcal{A}) = u_\beta(\mathcal{B})$.*

   (d) *for all $\beta \in [\omega \cdot \delta, \omega \cdot \delta + \omega)$ we have $u_\beta(\mathcal{A}) \geq u_\beta(\mathcal{B})$.*

In order to verify that we have an $\hat{\alpha}$-system, the following lemma — which, when combined with the preservation of atomic formulas, is called the back and forth property — will be important.

Lemma 2.2.15. *Suppose* $(\mathcal{A}, \overline{a}) \leq_\beta (\mathcal{B}, \overline{b})$. *Then for any* $\eta < \beta$ *and for any finite sequence* $\overline{d} \subseteq \mathcal{B}$ *there exists a sequence* $\overline{c} \subseteq \mathcal{A}$ *of equal length such that* $(\mathcal{B}, \overline{b}, \overline{d}) \leq_\eta (\mathcal{A}, \overline{a}, \overline{c})$.

Proof. Suppose that the conditions stated for $\leq_{2\cdot\delta}$ hold. Now suppose $\delta = \gamma + 1$. It suffices to show that for all finite sequences $\overline{d} \subseteq \mathcal{B}$ there exists a sequence $\overline{c} \subseteq \mathcal{A}$ of equal length such that $(\mathcal{B}, \overline{b}, \overline{d}) \leq_{2\cdot\delta+1} (\mathcal{A}, \overline{a}, \overline{c})$. We will extend f to $\overline{d}$ one element at a time. Let $d \in \overline{d}$, and suppose that $d \notin \langle \overline{b} \rangle$ (since if it were in that subgroup, we could simply map it to the corresponding element of $\langle \overline{c} \rangle$. Further suppose, without loss of generality, that $pd \in \langle \overline{b} \rangle$ and that $h(d) \geq h(d + s)$ for any $s \in \langle \overline{b} \rangle$. This last condition is often stated "d is proper with respect to $\langle \overline{b} \rangle$." These assumptions are reasonable, since if we need to extend f to an element farther afield, we can go one element at a time and work down to it. From this point, we essentially follow Kaplansky's proof of Ulm's theorem [39] to find the appropriate match for d. Use z to denote $f(pd)$. It now suffices to find some c of height $h(d)$ which is proper with respect to $\langle \overline{a} \rangle$ and such that $pc = z$.

First suppose that $h(z) = h(d) + 1$. Now both z and pd must be nonzero. For c we may choose any element of $(\mathcal{A})_{h(d)}$ with $pc = z$. The height of z tells us that there must exist such an element. We first check that $h(c) \leq h(d)$, which is easy, since if $h(c) > h(d)$, we would have

$$h(z) = h(pc) \geq h(c) + 1 \gneq h(d) + 1$$

Finally, it is necessary to show that c is proper with respect to $\langle \overline{a} \rangle$. Suppose that $c \in \langle \overline{a} \rangle$. Then $c = f(y)$ for some $y \in \langle \overline{b} \rangle$. Then $pd = py$ and $d - y \notin \langle \overline{b} \rangle$ to avoid $d \in \langle \overline{b} \rangle$. Further, $h(d - y) = h(d)$, since $h(y) = h(d)$ and d is proper with respect to $\langle \overline{b} \rangle$. However,

$$h(p(x - y)) = h(0) = \infty \gneq h(d) + 1$$

contradicting the maximality of $h(px)$. Thus $c \notin \langle \overline{a} \rangle$. Now suppose we have $h(c + t) \geq h(d) + 1$ for some $r \in \langle \overline{a} \rangle$ with $r = f(s)$. Since $c + r \neq 0$ (to avoid the case that $c = -r \in \langle \overline{a} \rangle$), we know that $h(p(w + r)) \geq h(d) + 2$, so that $h(p(d + s)) \geq h(d) + 2$. Since $h(r) \geq h(d)$, we also have $h(s) \geq h(d)$, so $h(d + s) = h(d)$, contradicting the maximality of $h(pd)$.

Suppose that $h(z) > h(d) + 1$. Now there is some $v \in (\mathcal{B})_{h(d)+1}$ such that $pd = pv$. Then the element $d - v$ is in $P_{h(d)}(\mathcal{B})$, has height $h(d)$, and is thus proper with respect to $\langle \overline{b} \rangle$.

Claim 2.2.16 (Lemma 13 of [39]). *Let the function*

$$r : (\langle \overline{b} \rangle_{h(d)} \cap p^{-1}(\mathcal{B})_{h(d+2)}) \to P_{h(d)}(\mathcal{B})$$

be defined as follows: For any $x \in (\langle \overline{b} \rangle_{h(d)} \cap p^{-1}(\mathcal{B})_{h(d)+2})$ there exists some $y \in (\mathcal{B})_{h(d)+1}$ such that $py = px$. Define Y by $Y : x \mapsto x - y$ and let $\hat{Y}$ be the composition of this map with the projection onto $P_{h(d)}(\mathcal{B})/P_{h(d)+1}(\mathcal{B})$. If

$$F : (\langle \overline{b} \rangle_{h(d)} \cap p^{-1}(\mathcal{B})_{h(d)+2})/\langle \overline{b} \rangle_{h(d)+1} \longrightarrow P_{h(d)}(\mathcal{B})/P_{h(d)+1}(\mathcal{B})$$

is the map induced by $\hat{Y}$ on the quotient, then the following are equivalent:

1. *The range of F is not all of $P_{h(d)}(\mathcal{B})/P_{h(d)+1}(\mathcal{B})$.*

2. *There exists in $P_{h(d)}(\mathcal{B})$ an element of height $h(d)$ which is proper with respect to $\langle \overline{b} \rangle$.*

Proof. To show that Condition 2 implies Condition 1, suppose $w \in P_{h(d)}$ has height $h(d)$ and is proper with respect to $\langle \overline{b} \rangle$. Then the coset of w is not in the range of F. Otherwise, $w = x - y + q$ for some $x \in \langle \overline{b} \rangle$, some $y \in (\mathcal{B})_{h(d)}$, and some $q \in P_{h(d)+1}(\mathcal{B})$. But then $h(w - x) > h(d)$, so w was not proper.

To show the other implication, suppose that w is an element of $P_{h(d)}(\mathcal{B})$ representing a coset not in the range of F. Then $h(w) = h(d)$. Further, w is proper, since if it were not, and if $h(s-w) > h(d)$ witnessed this, we could write $s - w = p\zeta$ with $\zeta \in (\mathcal{B})_{h(d)}$. But then $ps = p\zeta$ since $pw = 0$. But then F will map s to the coset of v, giving a contradiction. $\square$

Now since $d - v$ is such an element as is described in the second condition of the claim, we know that the range of F is not all of $P_{h(d)}(\mathcal{B})/P_{h(d)+1}(\mathcal{B})$. Since the vector spaces are finite (and thus finite dimensional), we know that the dimension of $(\langle\overline{b}\rangle_{h(d)} \cap p^{-1}(\mathcal{B})_{h(d)+2})/\langle\overline{b}\rangle_{h(d)+1}$ is less that $u_{h(d)}(\mathcal{B})$. However, since f was height preserving, it maps

$$(\langle\overline{b}\rangle_{h(d)} \cap p^{-1}(\mathcal{B})_{h(d)+2})/\langle\overline{b}\rangle_{h(d)+1}$$

$$\downarrow \text{onto}$$

$$(\langle\overline{a}\rangle_{h(d)} \cap p^{-1}(\mathcal{A})_{h(d)+2})/\langle\overline{a}\rangle_{h(d)+1}$$

Thus the dimension of $(\langle\overline{a}\rangle_{h(d)} \cap p^{-1}(\mathcal{A})_{h(d)+2})/\langle\overline{a}\rangle_{h(d)+1}$ is less than $u_{h(d)}(\mathcal{B})$.

In the case that $h(d) < \omega \cdot \delta + \omega$, we now know that the dimension of $(\langle\overline{a}\rangle_{h(d)} \cap p^{-1}(\mathcal{A})_{h(d)+2})/\langle\overline{a}\rangle_{h(d)+1}$ is less than $u_{h(d)}(\mathcal{A})$, so there is an element c_1 in $\mathcal{A}$ such that $pc_1 = 0$, $h(pc_1) = h(d)$, and which is proper with respect to $\langle\overline{a}\rangle$. Since $h(z) > h(d) + 1$, we may write $z = pc_2$ where $c_2 \in (\mathcal{B})_{h(d)+1}$. Now we write $c = c_1 + c_2$ and note that $pc = z$, that $h(c) = h(d)$, and finally that c is proper with respect to $\langle\overline{a}\rangle$.

If $h(d) \geq \omega \cdot \delta + \omega$, we need considerably less. In particular, it suffices to find some c such that $pc = z$, such that c is proper with respect to $\langle\overline{a}\rangle$, and such that $h(c) = \omega \cdot \delta + \omega$. This can be achieved by replacing $h(d)$ with $\omega \cdot \delta + \omega$ in the preceding argument, and noting that since $\omega \cdot \delta$ is a limit, $u_{\omega \cdot \delta} = \omega$. This completes the proof for the case $(\mathcal{A}, \overline{a}) \leq_{2 \cdot \delta} (\mathcal{B}, \overline{b})$ with δ a successor.

If δ is a limit ordinal, it suffices to consider some odd successor ordinal

$$2 \cdot \eta + 1 < 2 \cdot \delta$$

and to show that for any $\overline{d} \in \mathcal{B}$ there is some $\overline{c} \in \mathcal{A}$ such that

$$(\mathcal{B}, \overline{b}, \overline{d}) \leq_{2\cdot\eta+1} (\mathcal{A}, \overline{a}, \overline{c}).$$

Then the proof is exactly as in the successor case.

In the case that we start with $(\mathcal{A}, \overline{a}) \leq_{2\cdot\delta+1} (\mathcal{B}, \overline{b})$, we need to show that for any $\overline{d} \in \mathcal{B}$ there is some $\overline{c} \in \mathcal{A}$ such that $(\mathcal{B}, \overline{b}, \overline{d}) \leq_{2\cdot\delta} (\mathcal{A}, \overline{a}, \overline{c})$. Now we can follow the proof exactly as in the even successor case, except that we replace $\omega \cdot \delta + \omega$ with $\omega \cdot \delta$. $\square$

We now adapt the relations $\leq_\beta$ on pairs $(\mathcal{A}, \bar{a}), (\mathcal{B}, \bar{b})$ to relations on L.

Definition 2.2.17. *We say that* $(j_1, p_1) \leq_\beta (j_2, p_2)$ *if and only if*

$$(G^{j_1}, ran(p_1)) \leq_\beta (G^{j_2}, ran(p_2))$$

We need to verify that $(L, U, P, \hat{\ell}, E, (\leq_\beta)_{\beta<\hat{\alpha}})$ is an $\hat{\alpha}$-system. For the necessary effectiveness, notice that we need only consider $\leq_\beta$ on members of L, so only the groups G^i are considered. Conditions 1 – 3 are clear, as is the fact that $(\leq_\beta)_{\beta<\hat{\alpha}}$ is uniformly c.e. It remains to verify the following:

Lemma 2.2.18. *If* $\sigma u \in P$ *where* σ *ends in* ℓ^0 *and*

$$\ell^0 \leq_{\beta_0} \ell^1 \leq_{\beta_1} \cdots \leq_{\beta_{k-1}} \ell^k$$

where $\beta_0 > \beta_1 > \cdots > \beta_k$*, then there exists some* ℓ^* *such that* $\sigma u \ell^* \in P$ *and for all* $i \leq k$*, we have* $\ell^i \leq_{\beta_i} \ell^*$*.*

Proof. We write $\ell^i = (j_i, p_i)$. By Lemma 2.2.15, given $\ell^{k-1} \leq_{\beta_{k-1}} \ell^k$ we can produce an $\tilde{\ell}^{k-1} = (\tilde{j}_{k-1}, \tilde{p}_{k-1})$ such that $\tilde{p}$ extends p_{k-1} (mapping into the same structure) and $\ell^k \leq_{\beta_k} \tilde{\ell}^{k-1}$. Similarly, for each i, produce $\tilde{\ell}^i$ such that $\ell^{i+1} \leq_{\beta_{i+1}} \tilde{\ell}^i$. It will then be the case that for all i, $\ell^i \leq_{\beta_i} \tilde{\ell}^0$. If $u = 0$ or if 1 occurs somewhere in σ, let $\ell^* = (\tilde{j}_0, p^*)$, where p^* extends $\tilde{p}_0$ and its domain and range each contain the first n constants, where $2n+1$ is the length of σ. Now $\sigma u \ell^* \in P$ and for all i, $\ell^i \leq_{\beta_0} \ell^*$.

If, on the other hand, $u = 1$ and 1 does not occur in σ, then we may be sure that $\tilde{j}_0 = 0$. In this case, find some $j^* > 0$ such that $\alpha_{j^*} > \beta_0$. Note that since for each $\beta < \alpha_{j^*}$ we have $u_\beta(G^{j^*}) = u_\beta(G^0)$, it follows that

$$(G^{j^*}, \emptyset) \leq_{\beta_0+1} (G^0, \emptyset).$$

Thus, by Lemma 2.2.15, we have some sequence $ran(p^*) \subseteq G^{j^*}$ of length k such that $(G^0, ran(\tilde{p})) \leq_{\beta_0} (G^{j^*}, ran(p^*))$, where $2k+1$ is the length of σ. We define p^* to be the function taking each of an initial sequence of the natural numbers to the corresponding element of that sequence. Then clearly $\sigma u \ell^* \in P$, and for any i, we have $\ell^i \leq_{\beta_0} (G^0, ran(\tilde{p})) \leq_{\beta_0} (G^{j^*}, ran(p^*))$ $\square$

Now let S be an arbitrary $\Pi^0_{\hat{\alpha}}$ set. There is a $\Delta^0_{\hat{\alpha}}$ function $g(n, s) : \omega^2 \to 2$ such that for all n, we have $n \in S$ if and only if $\forall s[g(n, s) = 0]$, and such that for all $n, s \in \omega$, if $g(n, s) = 1$ then $g(n, s+1) = 1$. We define a $\Delta^0_{\hat{\alpha}}$ instruction function q_n as follows. If $\sigma \in P$ and σ is of length m, then we define $q_n(\sigma) = g(n, m)$.

Now we certainly can find computable and c.e. indices for all the components of the $\hat{\alpha}$-system (these indices do not vary with n), and uniformly in n we can find a $\Delta^0_{\hat{\alpha}}$ index for each q_n, so the Ash Metatheorem gives us (uniformly in n), a run π_n of (P, q_n) and the index for the c.e. set $\bigcup_{i \in \omega} E(\pi_n(2i))$. Let H^n denote the group whose diagram this is. Note that if $n \in S$, then $q_n(m) = 0$ for all m, and so $H^n \simeq G^0$. Otherwise there is some $\hat{m}$ such that for all $m > \hat{m}$, we have $q_n(m) = 1$, and so $H^n \simeq G^i$ for some $i \neq 0$. $\square$

2.3 Index Sets for Abelian p-Groups of Small Ulm Length

In all cases in the previous section, the completeness side of the result followed from the fact that the group with uniformly infinite Ulm invariant (up to the specified length) had an index set which was complete at the required level. We would like to know how exceptional this one structure is. For instance, might the group G with the same length but with $u_\alpha(G) = 1$ for all $\alpha < \lambda(G)$ have a simpler index set? The present section will document preliminary work toward this question and other related questions, which will appear in joint work of the present author with Harizanov, Knight, and S. Miller in [16].

In making these calculations for arbitrary Abelian p-groups of specified length, we give up many of the favorable conditions that made possible our use of the Ash Metatheorem, and have to fall back on the Khisamiev characterization. This limits most of our knowledge so far to the groups of length less than ω^2.

Proposition 2.3.1. *Let K be the class of reduced Abelian p-groups of length ωm, and let $\mathcal{A} \in K$. Then $I(\mathcal{A})$ is m-complete Π^0_{2m+1} within K.*

Actually, the case $m = 1$ was proved first in joint work of the present author with Cenzer, Harizanov, and Morozov, which is also still at a preliminary stage and will appear in [14].

Theorem 2.3.2. *Let K be the class of reduced Abelian p-groups of length $\omega M + N$ for some $M, N \in \omega$. Let $\mathcal{A} \in K$.*

1. *If for all n we have $u_{\omega M+n}(\mathcal{A}) < \infty$, then $I(\mathcal{A})$ is m-complete Π^0_{2M+1} within K.*
2. *If there is a unique $\hat{n} < N$ such that $u_{\omega M+\hat{n}}(\mathcal{A}) = \infty$ and for all $n' < \hat{n}$ we have $u_{\omega M+n'}(\mathcal{A}) = 0$, then $I(\mathcal{A})$ is m-complete Π^0_{2M+2} within K.*
3. *If there is a unique $\hat{n} < N$ such that $u_{\omega M+\hat{n}}(\mathcal{A}) = \infty$ and for some $n' < \hat{n}$ we have $0 < u_{\omega M+n'}(\mathcal{A}) < \infty$, then $I(\mathcal{A})$ is m-complete d-Σ^0_{2M+2} within K.*
4. *If there exist $n' < \hat{n} < N$ such that $u_{\omega M+n'}(\mathcal{A}) = u_{\omega M+\hat{n}}(\mathcal{A}) = \infty$ then $I(\mathcal{A})$ is m-complete Π^0_{2M+3} within K.*

The previous two results suffice for all Abelian p-groups of length less than ω^2. Some limited information is available on groups of greater length.

Theorem 2.3.3. *Let K be the class of reduced Abelian p-groups. Let $\mathcal{A}$ be a member of K with length greater than ωm. Then $I(\mathcal{A})$ is not Σ^0_{2m+2}*

2.4 The Isomorphism Problem for Torsion-Free Abelian Groups

The class of torsion-free Abelian groups has been important to the study of classification problems in descriptive set theory. Friedman and Stanley proved that the isomorphism relation on finite rank countable torsion-free Abelian groups is Borel (and thus not Borel complete) [25]. They also conjectured that isomorphism for arbitrary countable torsion-free Abelian groups was Borel complete. This conjecture has so far resisted proof, although partial results showing that it is not Borel have been achieved by Hjorth [37].

2.4.1 Finite Rank

In general, even finite rank torsion-free Abelian groups can have quite complicated structure. A group is of rank at least n if and only if there are at least n $\mathbb{Z}$-linearly independent elements (so in particular there are groups of finite rank that are not finitely generated; for example, the additive group of rationals has rank 1). Friedman and Stanley showed that the restriction of the isomorphism relation to the set of countable torsion-free Abelian groups of finite rank is not Borel complete [25]. There is a well-known classification for the case of rank 1, but none for any larger rank. Thomas has recently shown that the isomorphism relation on torsion-free Abelian groups of rank n is strictly simpler than on those of rank $n+1$ (in the sense of the ordering $\leq_B$; see Definition 4.2.1) [70]. As the Friedman-Stanley result suggests, though, these groups are not altogether intractable. It is helpful to note the following "normal form" result for these groups.

Lemma 2.4.1. *If G is a torsion-free Abelian group of rank $n < \omega$, then $G \leq \mathbb{Q}^n$.*

Proof. There is an embedding of G into $G_{\mathbb{Z}^*} = \{[\frac{1}{a}g] | a \in \mathbb{Z}^*, g \in G\}$, where the brackets denote classes of the natural equivalence relation. This group has the structure of a (necessarily free) $\mathbb{Q}$-module, given by $\frac{m}{n}g = [\frac{1}{n}mg]$. It is easy to show that $G_{\mathbb{Z}^*}$ has rank at most n. ☐

There is a known classification, due to Baer [6], of countable rank 1 torsion-free Abelian groups. The account here will generally follow that in the book by Fuchs [27]. Given such a group G, for any prime p, we define a function $h_p : G \to \mathbb{N}$ by setting $h_p(a)$ equal to the largest natural number k such that there is some $b \in G$ with $p^k b = a$. If no such k exists, we say $h_p(a) = \infty$. Now define the *characteristic* of a to be the sequence $\chi_G(a) = (h_{p_1}(a), h_{p_2}(a), \dots)$, where $(p_i)_{i \in \omega}$ is a list of all prime numbers. Where no confusion about the group involved is likely, we will write $\chi(a)$.

In some torsion-free Abelian groups (think of $(\mathbb{Q}, +)$), it is the case that all nonzero elements have the same characteristic. In these groups, we would need to look no further for invariants. However, in some others (for example, $(\mathbb{Z}, +)$) the characteristics of the various elements are essentially the same, but not identical. We say that two characteristics are equivalent if they are equal except in a finite number of places and in all places where they differ, both are finite. An equivalence class of characteristics under this relation is called a *type*. If $\chi_G(a)$ belongs to a type $\mathbf{t}$, then we say that $\mathbf{t}_G(a) = \mathbf{t}$, and that a is of type $\mathbf{t}$. A group G in which any two non-zero elements have the same type $\mathbf{t}$ is said to be homogeneous, and we say that $\mathbf{t}(G) = \mathbf{t}$ is the type of G. In particular, we note that any torsion-free Abelian group of rank 1 is homogeneous.

Proposition 2.4.2 (Baer [6]). *If G and H are torsion-free Abelian groups of rank 1, then $G \simeq H$ if and only if $\mathbf{t}(G) = \mathbf{t}(H)$.*

Proof. Suppose $g \in G$ and $h \in H$ are both nonzero. Write $\chi(g) = (m_1, m_2, m_3, \dots)$ and $\chi(h) = (n_1, n_2, n_3, \dots)$. Now since, as we have observed, G and H are homogeneous, if they are of the same type, then these two sequences differ at only finitely many places (say $i_1, \dots, i_t$), and both sequences are finite at all of these places. Now there is some $\tilde{g} \in G$ such that $p_{i_i} p_{i_2} \cdots p_{i_t} \tilde{g} = g$, and there is some $\tilde{h} \in H$ such that $p_{i_i} p_{i_2} \cdots p_{i_t} \tilde{h} = h$. Clearly $\chi(\tilde{g}) = \chi(\tilde{h})$. Thus, for any rational q, there is some $x \in G$ with $q\tilde{g} = x$ if and only if there is some $y \in H$ with $q\tilde{h} = y$. Since such x and y are necessarily unique (because the groups are torsion-free), and since every member of G is a rational multiple of $\tilde{g}$ (respectively with H and $\tilde{h}$), we obtain a bijection which is an isomorphism. ∎

A structure $\mathcal{A}$ is said to be computably categorical if for any two computable copies $\mathcal{A}_1 \simeq \mathcal{A}_2 \simeq \mathcal{A}$, there is a computable function witnessing $\mathcal{A}_1 \simeq \mathcal{A}_2$. It is well known that if G is a computable finite rank torsion-free Abelian group, then G is computably categorical (see, for instance, [21], [49], [58], [28]). To see this, note that if we have computable finite rank torsion-free Abelian groups G and H, a finite set $\overline{g}$ spanning G, and $h : G \to H$ an isomorphism, then we can pass effectively from computable indices for $(G, \overline{g})$ and $(H, h(\overline{g}))$ to an index for that same isomorphism as a computable function.

It is clear that if K is a class of computably categorical structures, then $E(K)$ must be at worst Σ^0_3 complete within K, since we can express "$\mathcal{A}_a$ and $\mathcal{A}_b$ are isomorphic" as "there exists an index for a computable function which is total, bijective, and respects the group operation," which is clearly a Σ^0_3 condition. Thus, we have a bound on the complexity of the isomorphism problem for finite rank torsion-free Abelian groups. This bound is sharp.

Theorem 2.4.3. *If K is the class of torsion-free Abelian groups of any fixed finite rank r, then $E(K)$ is Σ^0_3 complete within K.*

Proof. We have already shown that $E(K)$ is Σ_3^0 within K. We can reduce the completeness to a problem on computably enumerable sets. I am grateful to C. Jockusch for suggesting the proof of the following lemma.

Lemma 2.4.4. *If S is a Σ_3^0 set, then there exist sequences $(A_n)_{n\in\omega}$ and $(B_n)_{n\in\omega}$, uniformly computably enumerable, such that $A_n \triangle B_n$ is finite if and only if $n \in S$.*

Proof. It is well known that the set $Cof = \{e | W_e \text{ is cofinite}\}$ is Σ_3^0 complete (see, for instance, [68]). We could take $A_n = \omega$ and $B_n = W_n$ for all n. Then $A_n \triangle B_n$ is finite just in case $\omega \setminus B_n = \overline{W_n}$ is finite, which happens exactly when $n \in Cof$. □

Now from a computably enumerable set A we will pass effectively to $G(A)$, a computable torsion-free Abelian group of rank 1, in such a way that $G(A) \simeq G(B)$ if and only if $A \triangle B$ is finite. This will be done by making the characteristic of 1 in $G(A)$ (understood as a subgroup of $\mathbb{Q}$) equal to the characteristic function of A. If we can do this, then $\chi_{G(A)}(1)$ will differ from $\chi_{G(B)}(1)$ only in places indexed by members of $A \triangle B$, and in every place they are both finite.

Given an index for a computably enumerable set A, we can enumerate the set $C(A) = \{\frac{1}{p_e} | e \in A\}$, and thus we can also enumerate the subgroup of the rationals generated by $C(A)$. We can pad to pass to a computable copy of this same group, and we call the resulting group $G(A)$. Now $(G(A_n))_{n\in\omega}$ and $(G(B_n))_{n\in\omega}$ are uniformly computable sequences of groups such that $G(A_n) \simeq G(B_n)$ exactly when $n \in S$. □

2.4.2 Infinite Rank

The situation for infinite rank groups is somewhat more difficult. An analogue of Lemma 2.4.1 holds in this context, but these groups are not computably categorical. We will follow reasoning essentially due to Hjorth [37] to show that the isomorphism problem for this class is not hyperarithmetical, and so is properly Σ_1^1. However, there are many properly Σ_1^1 m-degrees which are not Σ_1^1 complete (the study of these degrees dates back at least to [23]; see [29] for more recent work), so it is not yet clear that the isomorphism problem for this class is Σ_1^1 complete.

The argument of Hjorth coded hereditarily countable sets (ranked in a natural way, so as to correspond to the various classes Π_α) into a certain kind of labeled graph, and then coded these graphs into the groups. Hjorth's method of encoding was almost exactly what is given here. The principal differences are in the substitution of the trees from Lemma 2.4.6 for hereditarily countable sets, and in a new proof of injectivity for the encoding, to replace Hjorth's more difficult proof. In place of hereditarily countable sets, we will use trees. It is not clear yet that all trees can be encoded in these groups. However, a certain class of trees can be encoded.

We will first make the following definition:

Definition 2.4.5. *Let x be a node in a tree. We define the rank of x, denoted $Rk(x)$ as follows:*

1. *We say that $Rk(x) = 0$ if and only if x is terminal.*

2. *We say that $Rk(x) = \beta > 0$ if and only if β is the supremum of the ranks of all successors of x.*

The important features of these trees is that each of them is well-founded and has computable rank function (i.e. a computable function assigning to each node x the computable ordinal $Rk(x)$), that each $\mathcal{E}_\alpha$, $\mathcal{A}_\alpha$, and $\mathcal{L}^\alpha$, as well as the T_n's approximating them, has rank α, and the trees $\mathcal{E}_\alpha$ (respectively, $\mathcal{A}_\alpha$ or $\mathcal{L}^\alpha_\infty$) have index sets which are m-complete Σ^0_α (respectively, Π^0_α) within the class of trees produced.

Proposition 2.4.6. *For any computable successor ordinal α, there exist computable trees $\mathcal{E}_\alpha$, $\mathcal{A}_\alpha$, and for any computable limit ordinal α, there exist computable trees $\mathcal{L}^\alpha_\infty$, and a sequence of trees $(\mathcal{L}^\alpha_k)_{k\in\omega}$ such that if S is a Π^0_α set,*

1. *If α is a successor ordinal, there is a uniformly computable sequence of trees with computable rank function $(\mathcal{T}_n)_{n\in\omega}$ such that for all n,*

$$\mathcal{T}_n \simeq \begin{cases} \mathcal{A}_\alpha & \textit{if } n \in S \\ \mathcal{E}_\alpha & \textit{otherwise} \end{cases}$$

2. *If α is a limit ordinal, there is a uniformly computable sequence of trees $(\mathcal{T}_n)_{n\in\omega}$ with computable rank function such that for all n,*

$$\mathcal{T}_n \simeq \begin{cases} \mathcal{L}^\alpha_\infty & \textit{if } n \in S \\ \mathcal{L}^\alpha_k & \textit{for some } k \textit{ otherwise} \end{cases}$$

The proof given here is due to Hirschfeldt and White [35], although the result was certainly known before them. For instance, the result was known to Ash [4] and to Goncharov, and probably many others. The proof by Hirschfeldt and White is used here because the trees produced are easy to work with. In this proof, we work below some (large) computable ordinal, and identify computable ordinals with their notations.

Proof. We will first describe the trees $\mathcal{A}_\alpha$, $\mathcal{E}_\alpha$, $\mathcal{L}^\alpha_\infty$, and $\mathcal{L}^\alpha_n$, and then indicate how to construct $\mathcal{T}_n$ to reflect some α, some n, and some Π^0_α predicate. First, take $\mathcal{A}_1$ to be the tree consisting of a root with a single successor, and $\mathcal{E}_1$ to be a root with infinitely many successors. We can define the rest of the trees inductively. For $\alpha = \beta + 2$, we will define $\mathcal{A}_\alpha$ to be a root node with infinitely many copies of $\mathcal{E}_{\alpha+1}$ attached, and $\mathcal{E}_{\alpha+2}$ to be a root node with infinitely many copies of each of $\mathcal{E}_{\alpha+1}$ and $\mathcal{A}_{\alpha+1}$ attached.

For a limit ordinal α, let $(\gamma_n)_{n\in\omega}$ be a fundamental sequence for α consisting only of successor ordinals. Since we will define $\mathcal{A}_{\lambda+1}$ and $\mathcal{E}_{\lambda+1}$ only in terms of trees for lower ordinals, there is no danger of circularity. We define $\mathcal{L}^\alpha_\infty$ to be a root with a copy of $\mathcal{A}_{\gamma_n}$ attached for each n. The tree $\mathcal{L}^\alpha_k$ will be the tree consisting of a root with a copy of $\mathcal{A}_{\gamma_n}$ attached for each $n \leq k$ and a copy of $\mathcal{E}_{\gamma_n}$ for each $n > k$.

Finally, we define the trees for $\alpha = \lambda + 1$ where λ is a limit ordinal. The tree $\mathcal{A}_\alpha$ consists of a root with infinitely many copies of $\mathcal{L}_n^\lambda$ attached for each $n \in \omega$. The tree $\mathcal{E}_\alpha$ is a root with infinitely many copies of $\mathcal{L}_n^\lambda$ for each $n \in \omega$ and infinitely many copies of $\mathcal{L}_\infty^\lambda$ attached.

We will now show that there are sequences $(\mathcal{T}_n)_{n\in\omega}$ as claimed. We will proceed by induction on α. For $\alpha = 1$, suppose $S \in \Pi_\alpha^0$ is defined by $\forall x R(x,n)$. Then the tree $\mathcal{T}_{n,0}$ will consist of a root labeled with 1, with one successor labeled with 0. At later stages, $\mathcal{T}_{n,s}$ will consist of $\mathcal{T}_{n,s-1}$ with one successor with label 0 to the root added for each stage $t < s$ such that $\exists x < tR(x,n)$. Define $\tilde{T}_n = \bigcup_{s\in\omega} \mathcal{T}_{n,s}$, and as usual apply padding to get a structure $\mathcal{T}_n$ with a computable universe. If $n \in S$, it is clear that we will never add a second successor to the root, and if $n \notin S$, we will have seen the counterexample by some stage $\hat{s}$ and will add more new successors in each stage thereafter.

For $\alpha = \beta + 2$, say that S is defined by $\forall x R(x,n)$ where R is a $\Sigma_{\beta+1}^0$ predicate. Now its complement, $\overline{R}(x,n)$ is a $\Pi_{\beta+1}^0$ predicate, so by induction, there is a uniformly computable sequence of trees

$$\mathcal{U}_{\beta+1,x,n} \simeq \begin{cases} \mathcal{A}_{\beta+1} & \text{if } \overline{R}(x,n) \\ \mathcal{E}_{\beta+1} & \text{otherwise} \end{cases}$$

Note that the notation of $\beta + 1$ is merely for bookkeeping and does not effect the isomorphism type of the tree. Now the tree $\mathcal{T}_n$ will consist of a root node with infinitely many copies of $\mathcal{E}_{\beta+1}$ attached, as well as infinitely many copies of $\mathcal{U}_{\beta+1,x,n}$ for each $x \in \omega$. Now if $n \in S$, then for no x will $\overline{R}(x,n)$ hold, so $\mathcal{T}_n$ consists of a root with infinitely many successors, each isomorphic to $\mathcal{E}_{\beta+1}$. If $n \notin S$, then for some $\hat{x}$ we have $R(x,n)$, so all of the copies of $\mathcal{U}_{\beta+1,x,n}$ will be isomorphic to $\mathcal{A}_{\beta+1}$, so $\mathcal{T}_n \simeq \mathcal{E}_\alpha$

For α a limit ordinal, say that S is defined as a computably enumerable conjunction of all $R_k(n)$ where R_k is a $\Pi^0_{\gamma_k}$ predicate. Recall that γ_k was a successor ordinal for all k. We define the predicate

$$C(m,n) = \bigvee_{k \leq m} R_k(n)$$

and since this is a $\Pi^0_{\gamma_m}$ predicate, we have, by induction, a sequence of trees

$$\mathcal{U}_{\gamma_m,m,n} \simeq \begin{cases} \mathcal{A}_{\gamma_m} & \text{if } C(m,n) \\ \mathcal{E}_{\gamma_m} & \text{otherwise} \end{cases}$$

So we define $\mathcal{T}_n$ to be the tree consisting of a root node with a single copy of $\mathcal{U}_{\gamma_m,m,n}$ attached for each $m \in \omega$. Then if $n \in S$, then all of the $R_k(n)$ hold, so all of the $C(m,n)$ hold, and so $\mathcal{T}_n$ is isomorphic to $\mathcal{L}^\alpha_\infty$. Otherwise, for some $\hat{m}$, we have $R_{\hat{m}}(n)$ fails, so for all $m > \hat{m}$ the predicate $C(m,n)$ also fails, so $\mathcal{T}_n \simeq \mathcal{L}^\alpha_{\hat{m}-1}$.

Finally, we consider $\alpha = \lambda + 1$ for some limit ordinal λ. Say S is defined by $\forall x R(x,n)$ where $R(x,n)$ is Σ^0_λ. Again we use $\overline{R}$ to denote the complement. By induction, there is a uniformly computable sequence of trees

$$\mathcal{U}_{\lambda,x,n} \simeq \begin{cases} \mathcal{L}^\lambda_\infty & \text{if } \overline{R}(x,n) \\ \mathcal{L}^\lambda_k & \text{for some } k \text{ otherwise} \end{cases}$$

Now we let $\mathcal{T}_n$ be a tree consisting of a root labeled $\alpha+1$ with infinitely many copies of $\mathcal{L}^\lambda_m$ attached for each $m \in \omega$, as well as infinitely many copies of $\mathcal{U}_{\lambda,x,n}$ for each $x \in \omega$. Again, the isomorphism type of $\mathcal{T}_n$ is as claimed. □

With this result in hand, we can proceed to prove something about the complexity of the isomorphism problem for computable infinite rank torsion-free Abelian groups. Now for each computable ordinal α, and for each Π^0_α set S, there is a uniformly computable sequence of trees $(\mathcal{T}_n)_{n\in\omega}$ with uniformly computable rank functions, which witness that $\mathcal{E}_\alpha$, $\mathcal{A}_\alpha$, and $\mathcal{L}^\alpha_\infty$ have the appropriate index set complexity.

Proposition 2.4.7. *If K is the class of computable infinite rank torsion-free Abelian groups, then there is no computable ordinal α such that $E(K)$ is Π^0_α within K.*

Proof. For each of the trees T we have just constructed, we will produce a corresponding computable group $G(T)$. This construction of $G(T)$ is entirely due to Hjorth. Fix a computable ordinal α, and fix computable sequences of primes, all distinct, $\{p_\beta\}_{0\leq\beta\leq\alpha}$, $\{\hat{q}_{\gamma,\beta}\}_{\gamma<\beta\leq\alpha}$, and $\{\hat{p}_{\gamma,\beta}\}_{\gamma<\beta\leq\alpha}$. Let S be a Π^0_α set, and fix $(\mathcal{T}_n)_{n\in\omega}$ with their rank functions as in Proposition 2.4.6.

Given a computable tree T, we will now form the labeled graph (V_T, E_T, f_T). We will suppress the dependence on T in the notation unless confusion is likely. Let V consist of all finite sequences $\langle A_1, \ldots, A_m\rangle$ such that A_i is a node in T and $rk(A_i) < rk(A_{i+1})$. These are the vertices of the graph. Define the edge set, E_n, to consist of all unordered pairs $\{\langle A_1, \ldots, A_m\rangle, \langle A_1, \ldots, A_m, A_{m+1}\rangle\}$ where both members of the pair are in V. Now we will color both the vertices and the edges of the graph with primes. Define the function f so that

$$f(\langle A_1, \ldots, A_m\rangle) = p_\beta \text{ where } rk(A_m) = \beta$$

and so that

$$f(\{\langle A_1, \ldots, A_m\rangle, \langle A_1, \ldots, A_m, A_{m+1}\rangle\}) = \begin{cases} \hat{q}_{\gamma,\beta} & \text{if } rk(A_m) = \beta, \\ & rk(A_{m+1}) = \gamma, \\ & \text{and } A_{m+1} \text{ is a} \\ & \text{successor of } A_m \\ \hat{p}_{\gamma,\beta} & \text{if } rk(A_m) = \beta, \\ & rk(A_{m+1}) = \gamma, \\ & \text{and } A_{m+1} \text{ is not a} \\ & \text{successor of } A_m \end{cases}$$

From this graph, we can now produce a group, $G(T)$. Consider first the set of all finite formal sums $\sum_{v \in V^0} q_v v$ where $V^0 \subseteq V$ and $q_v \in \mathbb{Q}$. This is clearly a torsion-free Abelian group of infinite rank, with the obvious addition. We define $G(T)$ to be the subgroup consisting of all elements of the form

$$\sum_{v \in V^0} \frac{k_v v}{f(v)^{l_v}} + \sum_{\{v,w\} \in E^0} \frac{n_{\{v,w\}}(v+w)}{f(\{v,w\})^{m_{\{v,w\}}}}$$

where $V^0 \subset V$ and $E^0 \subset E$ are both finite, $k_v, n_{\{v,w\}} \in \mathbb{Z}$, and $l_v, m_{\{v,w\}} \in \mathbb{N}$.

Roughly, the member of the group which consists of $1v$ where v is a vertex of the graph will represent that vertex, $(v+w)$ will represent the edge $\{v,w\}$, and we can calculate f by measuring the powers of various primes by which elements are divisible. This is the reason for including both $\hat{p}$ and $\hat{q}$ in the definition of the graph, so that we can identify when there is an edge, even if it does not represent a successor.

Since all of these constructions are computable from an index for the tree with its rank function, we can produce a uniformly computable sequence of groups $G_n = G(\mathcal{T}_n)$. We will now show that if α is a successor ordinal, $G_n \simeq G(\mathcal{A}_\alpha)$ if and only if $n \in S$, and if α is a limit ordinal, $G_n \simeq G(\mathcal{L}_\alpha^\infty)$ if and only if $n \in S$. For this purpose, it suffices to establish the following lemma.

Lemma 2.4.8. *Let T^1 and T^2 be trees. Then $T^1 \simeq T^2$ if and only if $G(T^1) \simeq G(T^2)$.*

Proof. Clearly an isomorphism of trees amounts to a renaming of the nodes of the tree, and thus to a renaming of the elements of vertices of the graph, and induces an isomorphism of labeled graphs. This in turn induces an isomorphism of groups.

The converse, as usual, is more difficult. We will prove it by constructing (not necessarily effectively) a reasonably canonical function π which will recover the tree from the group. In the definition that follows, we will define the term "β-good" just as Hjorth did, to refer roughly to elements in the group which will play the role of nodes in the reconstructed tree, and the function π, which will reconstruct the original tree.

Definition 2.4.9. *We define the notion of β-good and the function*

$$\pi : \{x \in G(T) | x \text{ is } \beta\text{-good for some } \beta\} \to \textit{Labeled Trees}$$

by induction on β.

1. *An element $g \in G(T)$ is said to be 0-good if and only if it is divisible by all powers of p_0. For any 0-good g, we define $\pi(g)$ to be the labeled tree consisting of one node with a marker r_g.*

2. *An element g is said to be β-good for $\beta > 0$ if and only if g is divisible by all powers of p_β and for any $\gamma < \beta$ and for any γ-good element h, there is $h' \in G(T)$ such that $\pi(h) \simeq \pi(h')$ (as unlabeled trees) and one of the following holds:*

   (a) $(g + h')$ is divisible by all powers of $\hat{q}_{\gamma,\beta}$

   (b) $(g + h')$ is divisible by all powers of $\hat{p}_{\gamma,\beta}$

   For any β-good g, we define $\pi(g)$ to be a tree consisting of a node labeled with r_g having, for each $\gamma < \beta$ and each γ-good element h with $(h + g)$ divisible by all powers of $\hat{q}_{\gamma,\beta}$, a successor which is the root (labeled r_h) of the tree $\pi(h)$. By this we mean that the entire tree $\pi(h)$ is included, with its root attached at the right place.

The remainder of the section will prove that from the isomorphism type of $G(T)$ we can recover the tree T. This part of the argument will be different from Hjorth's. The following statement is the correct form of the correspondence between tree nodes and β-good elements which makes the definition worthwhile.

Lemma 2.4.10. *An element $g \in G(T)$ is β-good if and only if there is*

$$v = \langle A_1, \ldots, A_m \rangle \in V \subseteq G(T)$$

and $s \in \mathbb{Q}$ such that $g = sv$ and such that $rk(A_m) = \beta$.

Proof. For $\beta = 0$, this follows clearly from the definitions of the group and of β-good. Suppose that the lemma is established for $\gamma < \beta$. Suppose there are s and v as stated. Then clearly, by the definition of the group, g is divisible by all powers of p_β. Also, if $\gamma < \beta$ and h is γ-good, then there are s_1, v_1 such that $s_1 \in \mathbb{Q}$ and $v_1 = \langle B_1, \ldots, B_{m_1} \rangle$ such that $h = s_1 v_1$ and $rk(B_{m_1}) = \gamma$. Now clearly $v_0 = \langle A_1, \ldots A_m, B_{m_1} \rangle$ is a member of V, since $\gamma < \beta$. Note that $\pi(v_0) = \pi(v_1)$. Also, $sv + sv_0$ is divisible by all powers of $\hat{q}_{\gamma,\beta}$ or $\hat{p}_{\gamma,\beta}$ according to whether B_{m+1} is a successor of A_m. Thus, g is β-good.

Conversely, suppose g is β-good. Then g is divisible by all powers of p_β. But by the definition of the group, the only such elements are the rational multiples of the vertices ending in nodes of rank β. ∎

Thus, given a group isomorphic to $G(T)$, we can recover something like the nodes of the tree purely from the group structure. Now an easy corollary allows us to find the element on which π will tell us the whole tree T.

Corollary 2.4.11. *Suppose all elements of T, including the root, are of rank at most α. Let x and y be α-good elements of $G(T)$. Then there is a rational number s such that $x = sy$.*

Proof. Apply the lemma to x and y to get $x = s_1 v_1$ and $y = s_2 v_2$. There is only one vertex ending in a node of rank α — namely the root — so we have $v_1 = v_2$. ∎

We will now prove that the function π does what was claimed.

Lemma 2.4.12. *Let $x \in G(T)$ be β-good, with $x = sv = s\langle A_1, \ldots, A_m \rangle$, as in Lemma 2.4.10. Then $\pi(x)$ is isomorphic to the substructure of T consisting only of nodes which extend A_m.*

Proof. We proceed by induction on β. For $\beta = 0$, the result is obvious. Suppose that the lemma is established for $\gamma < \beta$. Let $\{B^i\}_{i \in I}$ be the successors of A_m in T, with $rk(B^i) = \gamma_i < \beta$, and define $w_i = \langle A_1, \ldots, A_m, B^i \rangle$. Now clearly $v + w_i$ is divisible by all powers of $\hat{q}_{\gamma_i,\beta}$, and so is $sv + sw_i$. By the inductive assumption, $\pi(sw_i)$ is isomorphic to the substructure consisting only of nodes which extend B^i, so $\pi(x)$ contains an isomorphic copy of the tree below A_m in T.

To finish, it suffices to show that if r_h is a successor of r_x in $\pi(x)$, then $h = sw_i$ for some i. From the definition of π, we see that h is γ-good for some $\gamma < \beta$. We apply Lemma 2.4.10 to see $h = s_1 z$ for $s_1 \in \mathbb{Q}$ and $z \in V$. Since $(h + x)$ is divisible by all powers of $\hat{q}_{\gamma,\beta}$, we know that $\{v, z\} \in E$ and that if $z =< C_1, \ldots, C_{m_1} >$ then $m_1 = m + 1$, that for all $i \leq m$ we have $C_i = A_i$, and that C_{m_1} is a successor of A_m in T. Let $C_{m_1} = B_{\hat{i}}$. Then $h = s_1 w_i$, as desired. $\square$

Now it is clear that taking $\beta = \alpha$ in this lemma, we have established Lemma 2.4.8.

$\square$

Since we can complete this construction for any computable ordinal α, to show that $E(K)$ is not Π^0_α in K, it suffices to show that for some $\alpha_1 > \alpha$ we can compute any $\Pi^0_{\alpha_1}$ set from $E(K)$. $\square$

If the Friedman–Stanley conjecture is true, it seems likely that if K is the class of torsion-free Abelian groups of infinite rank, then $E(K)$ is m-complete Σ^1_1. If this completeness statement is true, then there must be a torsion-free Abelian group of non-computable Scott rank. Since for each computable ordinal α there is a computable $L_{\omega_1\omega}$ sentence which is true exactly of torsion-free Abelian groups of Scott rank at least α, we may conclude from Proposition 2.4.7, by Barwise compactness, that there is such a group with non-computable Scott rank. This fact may be thus be taken as evidence in favor of the Friedman-Stanley conjecture.

CHAPTER 3

ALGEBRAIC AND ALGORITHMIC PROPERTIES OF FIELDS

Some of the first work in computable model theory centered on computable fields. Notable advances were the work of Frölich and Sheperdson [26], Rabin [60] and Metakides and Nerode [52]. In more recent years, much of the work in computable structure theory has been devoted to developing general theory, and the easier examples with which to do this have tended to be graphs, linear orderings, Boolean algebras, vector spaces, and Abelian groups.

Today, model theory proper — as opposed to computable model theory — studies various classes of fields as some of its most productive examples (see, for instance, [34]), largely due to important applications of model theory to algebraic geometry and number theory. At the same time, algebraic geometers have taken an increasing interest in algorithmic aspects of the subject (see, for instance, [19]). Some recent work in the computable model theory of rings and fields — centered mainly on the existence of algorithms for ideal membership, splitting, and similar properties — is surveyed in a paper of Stoltenberg-Hansen and Tucker [69].

In this chapter, we will apply some of the techniques of the previous section to certain classes of fields. We will demonstrate the existence of fields of arbitrary Turing degree, and of fields with no Turing degree. We will show that the isomorphism problem for arbitrary fields of any fixed characteristic, and for real closed fields, is Σ^1_1 complete. We will also calculate the complexity of the isomorphism problem for

Archimedean real closed fields or the models of each of a certain class of strongly minimal theories, and the corresponding index sets.

The results of Section 3.1 are a part of a joint paper of the present author with Harizanov and Shlapentokh [17]. Those of Sections 3.3, and 3.5 are reported in [12].

3.1 Degrees of Fields

The isomorphism type of a field has something to say about what sorts of algorithmic structure it will admit. That is, the isomorphism type of a field $\mathcal{A}$ determines a set of Turing degrees $DI(\mathcal{A})$ in which $\mathcal{A}$ has copies.

One straightforward way in which to associate computability-theoretic information with algebraic information is to find a Turing degree that corresponds in some natural way to the *isomorphism type* of a given structure. A first attempt at this, suggested by Jochusch and explored by Richter [63], was to say that the *degree of* $\mathcal{A}$ is the least Turing degree in which $\mathcal{A}$ has some copy. However, Richter showed that not every structure has a degree in this sense.

Theorem 3.1.1 (Richter [63]). *For any Turing degree* $\mathbf{d}$*, there is an Abelian torsion group of degree* $\mathbf{d}$*. Moreover, there is an Abelian torsion group with no degree.*

A second approach, also suggested by Jockusch, is the so-called *jump degree* of a structure. For the αth jump degree of a structure $\mathcal{A}$, we take the least element among the αth jumps of degrees of copies of $\mathcal{A}$. Again, it is not certain that a given structure will have αth jump degree.

Theorem 3.1.2 (Oates [59]). *For any $\alpha \leq \omega$, there is an Abelian p-group G^α such that G^α has αth jump degree, but does not have βth jump degree for any $\beta < \alpha$.*

As we have seen, then, there are certain deficiencies to associating a *particular* degree to a structure. It is often more informative to associate a *set of degrees* with a structure. For a fixed countable structure $\mathcal{A}$, we write $DI(\mathcal{A})$ for the set of Turing degrees $\mathbf{d}$ such that $\mathcal{A}$ has a copy of degree $\mathbf{d}$. Knight showed that in all cases except the well-understood "trivial" case, this set must be closed upwards [42]. Richter's results stated above indicate that for some $\mathcal{A}$, the set $DI(\mathcal{A})$ does not have a least element. A general survey of properties of $DI(\mathcal{A})$ may be found in a paper by Knight [43]. In the present paper, examples will be given of particular kinds of structures $\mathcal{A}$ in which $DI(\mathcal{A})$ has prescribed properties.

After proving the results of this section, the present author became aware of other work by Harizanov and Shlapentokh which proves the same results, in addition to some others. The united project will be reported in [17].

Theorem 3.1.3. *Let $S \subseteq \omega$. Then there exists a field $\mathcal{A}_S$ such that $DI(\mathcal{A}_S)$ is exactly the set of degrees that compute enumerations of S.*

Proof. The proof will generally follow the method of Richter [63]. Let C_n designate the nth cyclotomic polynomial, and let ζ_n be a computable sequence such that ζ_n is a primitive p_nth root of unity, where p_n is the nth prime. Let $\mathcal{A}$ be the field $\mathbb{Q}(\{\zeta_n | n \in S\})$. Clearly, any copy of $\mathcal{A}_S$ computes an enumeration of S by simply looking to see which of the relevant roots appears.

Lemma 3.1.4. *Let $U \subseteq \omega$, such that S is c.e. in U. Then there is $F \simeq \mathcal{A}_S$ such that $F \leq_T U$.*

Proof. Let $\mathcal{A}^*$ be a computable prime algebraically closed field of characteristic 0. The set $\{\zeta_n | n \in S\}$ is a U-c.e. subset of $\mathcal{A}^*$, and so is the least subfield of $\mathcal{A}^*$ containing this set. We can use padding to produce a computable copy of this structure. ∎

Lemma 3.1.5. *The field $\mathcal{A}_S$ is not trivial.*

Proof. The result is obvious, but a proof is included here for completeness. Let U be a finite subset of $\mathcal{A}_S$. If it is empty, take $f : \omega \to \omega$ to be the permutation which exchanges $1_{\mathcal{A}_S}$ and $0_{\mathcal{A}_S}$ and fixes all other elements. Otherwise, consider some $x \in U$. Now U can contain only finitely many of $x, x+x, x+x+x, \ldots$, so let $\tilde{x}$ be the first such element which is not contained in U. Let f be the permutation of ω which exchanges $\tilde{x}+x$ and $\tilde{x}$ and fixes all other elements. In either case, f is a permutation fixing all elements of U which is not an automorphism of $\mathcal{A}_S$. □ □

Corollary 3.1.6. *For any set $S \subseteq \omega$, there is a field $\mathcal{A}^!_S$ such that*

$$DI(\mathcal{A}^!_S) = \{\mathbf{d} | S \leq_T \mathbf{d}\}.$$

Proof. Let $\mathcal{A}^!_S = \mathcal{A}_{S \oplus \bar{S}}$. The result follows. □

3.2 The Isomorphism Problem for Computable Fields

Intuition and experience tell us that the class of computable fields is quite complicated, perhaps so much that no classification could ever capture it. Previous work by Kudinov focused on existence of a computable "Friedberg enumeration".

Definition 3.2.1. *A Friedberg enumeration of K up to isomorphism is a list of numbers, each of which is an index for a member of K, such that each isomorphism type from K occurs exactly once in the list. The enumeration is said to be computable (or hyperarithmetical), when this list is.*

Goncharov and Knight had asked whether there was a computable Friedberg enumeration up to isomorphism of computable fields of fixed characteristic. Kudinov announced the following result:

Theorem 3.2.2 (Kudinov). *There is no computable Friedberg enumeration of the computable fields of characteristic 0.*

Knowing the complexity of the isomorphism problem for a class can tell us about the existence of Friedberg enumerations. The following is in [31]:

Proposition 3.2.3 (Goncharov-Knight). *If $I(K)$ is hyperarithmetical and there is a hyperarithmetical Friedberg enumeration of computable members of K up to isomorphism, then $E(K)$ is hyperarithmetical.*

The idea of the proof is that if $E(K)$ is Σ_1^1 (which it must always be) and there is a hyperarithmetical Friedberg enumeration of K up to isomorphism, then $E(K)$ is also Π_1^1.

We can prove the following:

Theorem 3.2.4. *When K is the class of computable fields of some fixed characteristic, $E(K)$ is m-complete Σ_1^1 within K.*

Then we have the following strengthening of Kudinov's result:

Corollary 3.2.5. *For any p, prime or zero, there is no hyperarithmetical Friedberg enumeration up to isomorphism of computable fields of characteristic p.*

The proof of Theorem 3.2.4 is by encoding graphs in fields. Friedman and Stanley [25] gave a method for doing this in a way which is well-defined and one-to-one on isomorphism types. What is needed beyond this is to show that if we start with a computable graph, we get a computable field. It is well known that the isomorphism problem for undirected graphs is Σ_1^1 complete (see [54], [57], or [61] for a proof.

Actually, Theorem 3.3.1 implies the characteristic 0 case of Theorem 3.2.4. This proof is certainly simpler. However, the proof in this section — including some points not fully described here — covers positive characteristic and stresses the relationship with Borel complexity. Also, it offers an opportunity to simplify, at least for characteristic 0, the difficult argument of the Friedman–Stanley paper.

3.2.1 Borel Completeness for Fields: The Friedman-Stanley Embedding

In 1989, Friedman and Stanley [25] showed that the class of countable fields of characteristic 0 is Borel complete, the maximal level of complexity in their sense. They proved this by constructing a Borel embedding from graphs into fields (a Borel embedding is a Borel measurable function which is well-defined and injective on isomorphism types).

Friedman and Stanley assume that they are given a graph whose connectedness relation is R. From this they construct a field. We will use $\overline{F}$ to indicate the algebraic closure of F, and (S) for the smallest field containing S.

Consider $\{e_i\}_{i\in\omega}$, algebraically independent over $\mathbb{Q}$. Let F_0 be the composite of all of the $\overline{(\mathbb{Q}(e_i))}$, and define the extension

$$L(R) = F_0(\{\sqrt{e_i + e_j} | iRj\})$$

To deal with positive characteristic, we could replace $\mathbb{Q}$ with $\mathbb{F}_p$ and the square root with some qth root where q is relatively prime to p, both in this construction and throughout the following argument. This function $L : \mathit{Graphs} \to \mathit{Fields}$ is both a Borel measurable function under the usual product topology [36], and well-defined on isomorphism classes. The difficulty is in showing that

Proposition 3.2.6. *L is injective on isomorphism classes.*

Proof. In particular, it is difficult to show that $\sqrt{e_m + e_n}$ cannot be expressed as a rational function of the various e_i and $\sqrt{e_j + e_k}$ where $\{j, k\} \neq \{m, n\}$. I will give here only the argument for characteristic 0, and will refer the reader to Shapiro's paper [65] for the more general argument. The main difficulties appear when we consider only the composite of $\overline{(\mathbb{Q}(e_i))}$ and $\overline{(\mathbb{Q}(e_j))}$ (where $i \neq j$), and ask whether it contains $\sqrt{e_i + e_j}$. I am grateful to W. Dwyer for the proof of the following lemma. A different proof is given in the paper by Friedman and Stanley [25], and there are others still by Abhyankar [1] and Shapiro [65]. In any case, proving the following lemma in positive characteristic is quite difficult.

Lemma 3.2.7. $\sqrt{e_i + e_j} \notin (\overline{\mathbb{Q}(e_i)} \cup \overline{\mathbb{Q}(e_j)})$, *where* $i \neq j$.

Proof. A polynomial p of degree n in $\overline{(\mathbb{Q}(e_i))}$ gives a branched n-sheeted covering where the fiber over any point a in $\overline{(\mathbb{Q}(e_i))}$ is the set of roots of $p = a$, and branch points represent the multiple roots (a Riemann surface). A continuous function to find the roots of the polynomial may be defined on this covering *with branch points deleted*, but not in any neighborhood including the branch points themselves. We first consider the possibility

$$\sqrt{e_i + e_j} = \sum_{k=1}^{n} a_k b_k$$

where a_k is algebraic over $\mathbb{Q}(e_i)$, and b_k is algebraic over $\mathbb{Q}(e_j)$. That is, $\sum\limits_{k=1}^{n} a_k b_k$ gives one of the square roots.

Now for simplicity we can say that there is a single polynomial $p_i(Z)$ over $\mathbb{Q}(e_i)$ of which all the a_k are roots, and similarly one p_j over $\mathbb{Q}(e_j)$ of which all the b_k are roots. We can view the composite field $(\overline{\mathbb{Q}(e_i)} \cup \overline{\mathbb{Q}(e_j)})$ as $\overline{\mathbb{Q}(e_i)} \times \overline{\mathbb{Q}(e_j)}$. Since p_i and p_j will each have only finitely many multiple roots and at most finitely many points at which the coefficients are not defined, we can define continuous functions giving a_k and b_k on the relevant covering spaces of

$$\overline{\mathbb{Q}(e_i)} \setminus \{\text{these finitely many "bad points" of } p_i, \text{ say } b_t\}$$

and

$$\overline{\mathbb{Q}(e_t)} \setminus \{\text{finitely many "bad points" of } p_j, \text{ say } f_t\}$$

Thus the expression $\sum\limits_{k=1}^{n} a_k b_k$ can be continuously defined on the relevant covering space of $\overline{\mathbb{Q}(e_i)} \setminus \{b_t\} \times \overline{\mathbb{Q}(e_j)} \setminus \{f_t\})$. We can view this as a plane with finitely many vertical and horizontal lines deleted. Since the multiple roots of $Z^2 = e_i + e_j$ lie along the antidiagonal $e_i + e_j = 0$, there is clearly a neighborhood in which we can define $\sum_{k=1}^{n} a_k b_k$ as a continuous function, but which contains points of the antidiagonal, so we cannot define $\sqrt{e_i + e_j}$ as a continuous function. Thus the two cannot be equal. To make the difference more transparent, we could say that anywhere on this neighborhood we stay on the same branch of the right-hand side, but move from one branch to another on the left-hand side.

In the more general case that

$$\sqrt{e_i + e_j} = \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} c_k d_k}$$

we could simply write

$$(\sqrt{e_i + e_j})(\sum_{k=1}^{n} c_k d_k) = \sum_{k=1}^{n} a_k b_k$$

and again the right-hand side can be defined continuously where the left-hand side cannot. $\square$

There does not seem to be a way to modify this proof to cover the positive characteristic case. There is no apparent topology to replace the metric topology on the affine space, which we used here in declaring functions continuous or not. Also, while we could talk about the number of values of the "root function" and the right-hand-side "function," there are points in the proof at which it is not obvious that values will not collapse.

If we simply add more e_i, then more dimensions are added to the picture, but nothing really changes, since the diagonal for $\sqrt{e_i + e_j}$ is still in the same plane, and we can still find a neighborhood containing some point of it which contains no point of any line parallel to an axis. The next real problem comes up when we allow some square roots to be added. To simplify the task of visualization, and also to simplify the notation necessary, we will restrict the geometrical argument to a space whose F_0-dimension is the least possible to account for all e_i used in the expression. This allows us to refer to codimension, allowing an economical way to describe the higher-dimensional generalizations of the fact that lines intersect in points, planes intersect in lines, and so forth.

Lemma 3.2.8. *Let F_0 be as above. Then*

$$\sqrt{e_i + e_j} \notin F_0(\sqrt{e_j + e_k})$$

where i, j, and k are distinct.

Proof. Suppose not. First we will suppose again the simpler case where

$$\sqrt{e_i + e_j} = \sum_{s=1}^{n} \prod_{t=1}^{m_s} a_{st}$$

where each a_{st} is algebraic over a single e_q or is $f_{st}\sqrt{e_j + e_k}$ for some $f_{st} \in F_0$. Since there are only a finite number of such e_q involved in the expression, let us collect, as before, polynomials p_q, one to account for all a_{st} algebraic over a single e_q. The multiple roots of p_q may be collected, as before, as $\{b_{q\gamma}\}$. Those used in the composition of f_{st} may be collected as $\{d_{st\gamma}\}$. The multiple roots corresponding to $\sqrt{e_i + e_j}$ still form the diagonal (now a hyperplane, i.e. an algebraic surface of codimension 1) $e_i + e_j = 0$. Let $\hat{x}$ be a point of $e_i + e_j = 0$, and let N be a ball around it of positive radius. Use M_δ to denote the (finitely many) hyperplanes $X_q = b_{q\gamma}$, $e_t = d_{st\gamma}$, and $e_j + e_k = 0$. Now M_δ^c (the complement of M_δ) is open, so $N \cap (\bigcap_\delta M_\delta)$ is a neighborhood containing a point of $e_i + e_j = 0$ and no point of any M_δ. Thus, there is a neighborhood in which we stay on a single branch of the right-hand side of the supposed equation, but cross a branch point of the left hand side.

Just as before, the extension to the more general case,

$$\sqrt{e_i + e_j} = \frac{\sum_{s=1}^{n} \prod_{t=1}^{m_s} a_{st}}{\sum_{s=1}^{n} \prod_{t=1}^{m_s} b_{st}}$$

is quite easy. We clear the denominator and still have regions which are entirely fine for the right side of the equation but that the left finds unmanageable. $\square$

Alternately, we could consider a homomorphism

$$F_0(\sqrt{e_j + e_k}) \to F_0(\sqrt{e_j + e_k})$$

which sends $e_k \mapsto 0$ but which is the identity on $\mathbb{Q} \cup \{e_\ell\}_{\ell \neq k}$. If the lemma failed, this homomorphism would show that $\sqrt{e_i + e_j} \in F_0$, contrary to the previous lemma. A similar alternate proof is possible for the following lemma.

Similarly, one can establish

Lemma 3.2.9. *Let F_0 be as above. Then $\sqrt{e_i + e_j} \notin F_0(\{\sqrt{e_m + e_n} | \{m, n\} \neq \{i, j\}\})$.*

Proof. Suppose that the lemma fails. Then we suppose

$$\sqrt{e_i + e_j} = \sum_{s=1}^{n} \prod_{t=1}^{m_s} a_{st}$$

where each a_{st} is algebraic over a single e_q or is $f_{st}\sqrt{e_n + e_m}$ for some $f_{st} \in F_0$ and some $\{m, n\} \neq \{i, j\}$. Acting just as before, we denote by p_q the polynomial accounting for all a_{st} which are roots of some polynomial over e_q. The left-hand side of the equation still gives us multiple roots along the hyperplane $e_i + e_j = 0$, the roots of the p_q still form hyperplanes parallel to the axes, just as before. The only difference from the previous case is that there are more hyperplanes of diagonal type ($e_m + e_n = 0$), but there is still a neighborhood in which the right side of the equation works, and the left does not. By this point, the usual extension to the more general form of a member of

$$F_0(\{\sqrt{e_m + e_n} | \{m, n\} \neq \{i, j\}\})$$

is obvious. $\square$

We will need an additional fact. If two fields of this kind are isomorphic, the isomorphism will move e_i to something interalgebraic with some e_j, since they are the elements whose algebraic closure is included. The change of e_i to e_j is clearly tolerable, since it merely corresponds to a permutation of the names of the vertices of a graph. However, we need to verify that the isomorphism does not foul up information on the connectedness relation.

Lemma 3.2.10. *Let $A \sim B$ if and only if $\overline{\mathbb{Q}(A)} = \overline{\mathbb{Q}(B)}$. If $e \sim e_i$ and $c \sim e_j$ then $\sqrt{e+c} \notin (\overline{\mathbb{Q}(e_i)} \cup \overline{\mathbb{Q}(e_j)})$.*

Proof. The proof of this lemma is trivial. If $\sqrt{e+c} \in (\overline{\mathbb{Q}(e_i)} \cup \overline{\mathbb{Q}(e_j)})$, then it is also in the (exactly equal) set $(\overline{\mathbb{Q}(e)} \cup \overline{\mathbb{Q}(c)})$, in contradiction to the previous lemma. □

Lemma 3.2.11. *If $c_i \sim e_i$ for all i, then*

$$\sqrt{c_i + c_j} \notin F_0(\{\sqrt{e_m + e_n} | \{m,n\} \neq \{i,j\}\})$$

Proof. This proof is an almost equally obvious extension of earlier results. Let F_1 denote the composite field of all the $\overline{\mathbb{Q}(c_i)}$. The field

$$F_0(\{\sqrt{e_m + e_n} | \{m,n\} \neq \{i,j\}\})$$

is equal to the field

$$F_1(\{\sqrt{e_m + e_n} | \{m,n\} \neq \{i,j\}\})$$

We should note that each e_i is the root of some polynomial over c_i. Now suppose

$$\sqrt{c_i + c_j} = \sum_{s=1}^{n} \prod_{t=1}^{m_s} a_{st}$$

where each a_{st} is algebraic over a single c_i or is $f_{st}\sqrt{e_m + e_n}$ for some $f_{st} \in F_1$ and some $\{m,n\} \neq \{i,j\}$. Let p_q again denote the polynomial accounting for all a_{st} algebraic over c_q. The left-hand side still gives the same diagonal hyperplane $c_i + c_j = 0$. On the right-hand side, we have a finite union of hyperplanes parallel to the axes (the multiple roots of p_q), and also some more exotic hypersurfaces. These hypersurfaces are those of the form $e_m + e_n = 0$. However, these are not equal to $c_i + c_j = 0$, so for each such hypersurface P (of only finitely many) there is some neighborhood containing a point of $c_i + c_j = 0$ but no point of P. Thus, we can still find the necessary neighborhood in which the right-hand side of the equation is continuous and the left-hand side is not. The more general element works as always. $\square$

We can now prove Proposition 3.2.6. Suppose that R and S are two graphs, and that $L(R) \simeq L(S)$. Now by this isomorphism, each $e_i \in L(R)$ is mapped to some $c_i \sim e_j \in L(S)$. Certainly if nRm then $\sqrt{e_n + e_m} \in L(R)$ and thus $\sqrt{c_n + c_m} \in L(S)$. By the last lemma, if $c_n \sim e_p$ and $c_M \sim e_q$, the last statement implies that $\sqrt{e_p + e_q} \in L(S)$, so by the previous lemma pSq (since $\sqrt{e_p + e_q} \notin F_0(\{\sqrt{e_m + e_n} | \{m, n\} \neq \{i, j\}\})$ (that is, $e_p + e_q$ only had a square root if we put one in to account for a connection of p and q in S). Similarly, we can argue that if nSm, then the corresponding elements are connected in R. Thus $R \simeq S$. $\square$

3.2.2 Computable Construction of the Friedman-Stanley Embedding

It will turn out that a similar embedding produces computable fields from computable graphs, amounting to a reduction $E(\text{Graphs}) \leq_1 E(\text{Fields})$. This will complete the proof. The only real modification necessary is to guarantee that if we start with a computable graph, we end up with a computable field.

We should note that since the class of fields of given characteristic has $\prod_2^0$ axioms (stating that it is a commutative ring, plus the condition that for any element there exists a multiplicative inverse), $I(K)$ is $\prod_2^0$. Given a computable directed graph with connectedness relation R, consider $\{e_i\}_{i\in\omega}$, algebraically independent over $\mathbb{Q}$. Let G_0 be a computable field isomorphic to the composite of all of the $\overline{\mathbb{Q}(e_i)}$, and let $M(R)$ be the extension $G_0(\{\sqrt{e_i + e_j} | iRj\})$. It remains to verify

Proposition 3.2.12. *Given a graph R we get a field $F(G)$ computable relative to G.*

Proof. Consider the language of fields, plus countably many constants e_i, with the theory of algebraically closed fields of characteristic 0 and the sentences stating that the e_i are algebraically independent. This theory is complete and decidable (since the theory of algebraically closed fields alone proves quantifier elimination), and so it has a computable model. Call this computable model G.

Once we have G, there is a c.e. $G_0^* \subseteq G$ which contains exactly those members of G which are algebraic over a single e_i. Further, there is a c.e. $R^* \subseteq G$ consisting of exactly $\{\sqrt{e_i + e_j} | iRj\}$. With these two sets, we can enumerate the elements of the smallest subfield containing $G_0^* \cup R^*$, and we will call this $F^* \subseteq G$. Note that F^* has c.e. universe. Let $idx(R)$ be the index of the function with which we enumerate $D(F^*)$, and note that we can find it effectively in a uniform way from an index for $D(R)$. Now by padding, we can replace the c.e. field F^* by a field whose universe is computable. It is also clear that an index for the function with which we enumerate this field is effectively obtained from $idx(R)$ in a uniform way. This completes the proof both of the proposition and of Theorem 3.2.4 in the case of characteristic zero. □

3.3 The Isomorphism Problem for Computable Real Closed Fields

Arbitrary fields, then, are quite complicated. On the other hand, many natural classes of fields are more tangible, and might give better hope of classification. K. Manders suggested the example of real closed fields, whose model theory is reasonably well-behaved, but which is unstable. Recall that a real closed ordered field is an ordered field satisfying the additional condition that each odd-degree polynomial has a root. When we add positive infinite elements we have a great deal of freedom in the structure of the field.

Theorem 3.3.1. *If K is the class of real closed ordered fields, then $E(K)$ is Σ^1_1 complete within K.*

Proof. We will say that $a \preceq b$ exactly when $a \leq b^n$ for some $n \in \omega$. We say that $a \approx b$ (a is comparable to b) if $a \preceq b$ and $b \preceq a$. Notice that the $\sim$-classes of positive elements form intervals. We will write $a \prec b$ if $a \preceq b$ but it is not the case that $b \preceq a$. The proof will depend on realizing an arbitrary computable linear order as the order type of the comparability classes of infinite elements.

Lemma 3.3.2. *Given a computable linear order L, there is a computable structure $R(L)^* = (\bar{R}, +, \cdot, 0, 1, \leq, \{e_i\}_{i \in L})$, an expansion of a real closed field, in which $e_i \preceq e_j$ if and only if $i \leq j$. Moreover, an index for $R(L)^*$ is computable from an index for L.*

Consider the language of ordered fields, plus infinitely many constants e_i, with the theory of real closed fields, and the sentences for each i stating that e_i is greater than any polynomial in $\{e_j | j <_L i\}$, and that all are greater than polynomials in 1. This is a complete, decidable theory, and thus has a computable model G. There is a c.e. subset $\tilde{R} \subseteq G$ containing exactly the elements algebraic over $\{e_i\}_{i \in \omega}$. From an index for L, we can effectively find an index $i(L)$ for the function enumerating $\tilde{R}$. Again we can pad to find an isomorphic structure $R(L)^*$ with computable universe, as claimed. Let $R(L)$ denote the reduct of $R(L)^*$ to the language of ordered fields.

We have encoded arbitrary linear orders into real closed fields, and all that remains is to make sure that this operation is well-defined and injective on isomorphism types. The well-definedness is clear, since an isomorphism of linear orders would just amount to a permutation of the labels for the e_j. It is also clear that if

h is an isomorphism $h : R(L_1) \to R(L_2)$, then for $a, b \in R(L_1)$, $a \preceq b$ if and only if $h(a) \preceq h(b)$, but it requires some verification to see that for a in the comparability class of some e_i, $h(a)$ must be in the comparability class of some e_j. Once this is shown, h will induce an isomorphism of orders $\tilde{h}$, where if h maps the class of e_i to that of e_j, then $\tilde{h} : i \mapsto j$. I am grateful to L. van den Dries for suggesting the proof of the following lemma.

Lemma 3.3.3. *Let F be $R(L)$ for some linear order L. Let C be a positive infinite comparability class of elements of F. Then C is the comparability class of one of the e_i.*

Proof. Suppose we have a real closed field K, and we add a single positive infinite element $x > K$. Let $K((e^{\mathbb{Q}}))$ denote the set of formal series $f = \sum_{q \in \mathbb{Q}} a_q e^q$, where $a_q \in K$ and $a_q = 0$ except for q in some well-ordered set. There is an isomorphism $rcl(K(x)) \simeq K((e^{\mathbb{Q}}))$ mapping $x \mapsto e^{-1}$. Now suppose that $y \in K((e^{\mathbb{Q}}))$, and $y = \sum_{q \in \mathbb{Q}} b_q e^q$. Further, suppose that for all $x \in K$, we have $y > x$ (that is, y is an infinite element over K). Let $\hat{q}$ be the least such that $b_{\hat{q}} \neq 0$. We know that $\hat{q} < 0$, since if $\hat{q} \geq 0$, then $y \leq b_0 + 1$, but $b_0 + 1 \in K$, giving a contradiction. Now $e^{\hat{q}-1} > y$; that is, $(e^{-1})^{1-\hat{q}} > y$, so $y \approx t^{-1}$. Thus, $rcl(K(x))$ has exactly one more comparability class than K.

Given this, the lemma is relatively easy. Using the previous paragraph as an induction step, it is easy to show that for L a finite linear order, the lemma holds. Further, since any element in $R(L)$ is algebraic over finitely many e_i, $R(L) = \bigcup_I rcl(R(\{e_i\}_{i \in I}))$ where I is a finite subset of L. This completes the proof of both the lemma and the theorem. ∎ ∎

Corollary 3.3.4. *If K is the class of countable real closed ordered fields and K' is a class of countable structures, then $K' \leq_B K$ (see Definition 4.2.1)*

It is also worthwhile to note that Theorem 3.3.1 implies the characteristic 0 case of Theorem 3.2.4.

3.4 The Isomorphism Problem and Index Sets for Computable Models of Certain Strongly Minimal Theories

In a recent paper of Goncharov, Harizanov, Laskowski, Lempp, and McCoy [30], it was shown that in a certain sense it is impossible to code complicated sets into countable models of certain uncountably categorical theories. The authors were interested in coding the information into an individual model, and proved that this was impossible for trivial strongly minimal theories (specifically, any model has a $\emptyset''$-decidable copy). The current section asks a similar question, but in a much different sense. Inspired by the results of the previous section, we will demonstrate a sense in which complicated information cannot be encoded in the *class* of computable models.

Much depends on the work of Baldwin and Lachlan [8], Morley [53], and Marsh [51]. Morley showed that any theory categorical in a single uncountable cardinal was categorical in all uncountable cardinals. Baldwin and Lachlan showed that such a theory must have either one or $\aleph_0$ countable models, and Harrington [32] and Khisamiev [40], working independently, showed that if the theory is decidable, then all of its models have decidable copies.

Here we focus on the simplest such theories, the strongly minimal ones. A theory is said to be *strongly minimal* if every subset of the model which is definable (with parameters) is either finite or co-finite. We will seek to show that such theories not only have tight control over their countable models, but also over the computable ones. Marker's book [50] is a helpful reference for much of the needed model-theoretic background.

The isomorphism problem result of this section is a more general statement of analogous theorems for vector spaces and algebraically closed fields proved in [12]. The index set results are generalized from analogous results for vector spaces in joint work of the present author with Harizanov, Knight, and S. Miller [16].

If we have an indexed set $\{a_i\}_{i\in\omega}$ and $R \subseteq \omega$, then we will write $\bar{a}_R$ for $\{a_i | i \in R\}$. We will also write $P(S)$ for the algebraic closure of the set S.

Theorem 3.4.1. *Let T be a strongly minimal theory with effective elimination of quantifiers and some computable model, where $acl(\emptyset)$ is infinite. Suppose K is the class of computable models of T. Then $E(K)$ is m-complete Π^0_3 within $I(K)$.*

Proof. It suffices that we can define the relation "$dim(\mathcal{A}_i) = dim(\mathcal{A}_j)$" by a Π^0_3 statement. In what follows, when we have a sequence $v_1, \ldots, v_n$, we will write $\hat{v}_i$ for the sequence with v_i omitted, and if ϕ is a formula, we will understand $\phi(\hat{v}_i, z)$ to mean $\phi(v_1, \ldots, v_{i-1}, z, v_{i+1}, \ldots, v_n)$. If $\mathcal{M}$ is a model of T, we can write that $b_1, \ldots, b_n$ are independent members of $\mathcal{M}$ as the formula

$$\bigwedge_{i=1}^{n} \mathop{\bigwedge\!\!\!\bigwedge}_{\substack{\psi(\overline{v}, x)\\ m\in\omega}} \left(\left[\exists x_1, \ldots, x_m \forall y \left[\psi(\hat{b}_i, y) \rightarrow \bigvee_{j=1}^{m} x_j = y \right] \right] \rightarrow \neg\psi(\hat{b}_i, b_i) \right).$$

We will call this conjunction $I_n(b_1, \ldots, b_n)$. Note that because of the effective quantifier elimination, this is equivalent to a computable Π_1 sentence.

Now to write that a model has dimension at least n, we would write the sentence

$$D_n = \exists x_1, \ldots, x_n I(x_1, \ldots, x_n)$$

which is Σ^0_2. To define the isomorphism relation within K, we need only write

$$\mathop{\bigwedge\!\!\!\bigwedge}_{n\in\omega} \mathcal{A}_a \models D_n \iff \mathcal{A}_b \models D_n$$

This condition is clearly Π^0_3, and its restriction to the set of ordered pairs of indices for models of T is clearly $E(K)$, since it says that $\varphi(\overline{a}, \mathcal{A}_a)$ and $\varphi(\overline{a}, \mathcal{A}_b)$ have the same dimension.

Toward completeness, note that $cof = \{e|W_e \text{ is cofinite}\}$ is m-complete Σ^0_3. We will produce a uniformly computable sequence of models $(\mathcal{A}_n)_{n\in\omega}$ such that $\mathcal{A}_n$ has dimension equal to $|\overline{W}_n|$. When this is accomplished, we will have reduced the complement of cof to the index set of the infinite dimensional model of T. Let B be a copy of ω on which we will build the structure $\mathcal{A}_n$, and let $f_{-1} = D_{-1} = \emptyset$. We will build $\mathcal{A}_n$ by giving a function mapping B to some substructure of $\mathcal{A}^*$, an infinite-dimensional model with distinguished computable basis $\{a_i|i \in \omega\}$. In particular, $\mathcal{A}_n$ will be isomorphic via f to the substructure of $\mathcal{A}^*$ with basis $\{a_i|i \notin W_n\}$.

We begin by defining our estimates for $S = \omega - W_n$. Let $S_0 = \emptyset$. At stage s, if $S_{s-1} \subseteq \omega - W_{n,s-1}$, then we find the furst element x outside $W_{n,s}$ and let $S_s = S_{s-1} \cup \{x\}$. On the other hand, if $S_{s-1} \not\subseteq \omega - W_{n,s-1}$, then we find the least $r \in S_s \cap W_{n,s}$ and set $S_s = S_{s-1} \cap \{q|q < r\}$. Now to build $\mathcal{A}_n$, we will proceed as follows.

At each stage s where $S_{s-1} \not\subseteq S_s$, we will have to collapse to a smaller model. There are some elements in $ran(f_{s-1})$ which should not be in $ran(f)$. Let t be the last stage less than s such that $S_t \subseteq S_s$. Now f_{s-1} maps some elements $\bar{d}$ of B into $P(\bar{a}_{S_t})_s$, and some other elements $c_1, \ldots, c_j$ to $P(\bar{a}_{S_s})_s - P(\bar{a}_{S_t})_s$. Let $\delta_s(\bar{d}, c_1, \ldots, c_j)$ be the conjunction of all elements of D_{s-1}. Now $P(\bar{a}_{S_{s-1}}) \models \exists \bar{x}\ \delta_s(\bar{d}, \bar{x})$, and so we also have $P(\bar{a}_{S_s}) \models \exists \bar{x}\ \delta_s(\bar{d}, \bar{x})$. We run the enumeration until we see the witnesses $\tilde{c}_1, \ldots, \tilde{c}_j$ such that $P(\bar{a}_{S_s}) \models \delta_s(\bar{d}, \tilde{c}_1, \ldots, \tilde{c}_j)$. We now define f_s so that $f_s \supseteq f_t$, so that $f_s(c_i) = \tilde{c}_i$, and so that $dom(f_s)$ includes the first s elements of B and $ran(f_s)$ includes the first s elements of $P(\bar{a}_{S_s})$.

Otherwise, no such collapse is necessary. We simply extend f_{s-1} to f_s so that $dom(f_s)$ includes the first s elements of B and $ran(f_s)$ includes the first s elements of $P(\bar{a}_{S_s})$.

In any case, we also extend D_{s-1} to D_s by finding the first atomic sentence λ

such that neither λ nor $\neg\lambda$ is in D_{s-1} but all constants used in λ are in $dom(f_s)$, computing its truth value via f_s, and recording the result in D_s.

A *true stage* is a stage s such that $S_s \subseteq \omega - W_n$.

Lemma 3.4.2. *There are infinitely many true stages.*

Proof. Consider some non-true stage s. Suppose that r^* is the least element of S_s which is not in $\omega - W_n$. There is some least stage $t \geq s$ at which $r^* \in W_{n,t}$. Now $t + 1$ will be a true stage. $\square$

Let $\mathcal{A}_n$ be the structure with atomic diagram $\bigcup\limits_{s\in\omega} D_s$, and universe B. Let T be the set of all true stages, and let $f = \bigcup\limits_{s\in T} f_s$. Now $\mathcal{A}_n$ is isomorphic to a substructure of $\mathcal{A}^*$ via f. $\square$

This has actually proved a first index set result.

Corollary 3.4.3. *Let T be a strongly minimal theory with effective elimination of quantifiers and a computable model, where $acl(\emptyset)$ is infinite, and let $\mathcal{M}$ be its infinite dimensional computable model. Then if K is the class of computable models of T, then $I(\mathcal{M})$ is m-complete Π^0_3 within K.*

We can also compute the complexities of the index sets for the other computable models of T.

Theorem 3.4.4. *Let T be a strongly minimal theory with effective elimination of quantifiers and a computable prime model, where $acl(\emptyset)$ is infinite, and let $\mathcal{M}_0$ be the prime model of T. If K_f is the class of computable finite-dimensional models of T, then $I(\mathcal{M}_0)$ is m-complete Π^0_2 within K_f.*

Proof. The complexity may be bounded using the methods of Theorem 3.4.1. In particular, the copies of $\mathcal{M}_0$ are those models of T that satisfy the sentence

$$\forall x_1, x_2\ \neg I_2(x_1, x_2).$$

Toward completeness, let $\mathcal{M}_0 \prec \mathcal{M}_1$ where $\mathcal{M}_1$ has dimension 1. Consider an arbitrary Π^0_2 set $S = \{n | \exists^\infty x\ R(n,x)\}$. We will build a uniformly computable sequence of structures $(\mathcal{A}_n)_{n\in\omega}$ such that

$$\mathcal{A}_n \simeq \begin{cases} \mathcal{M}_0 & \text{if } n \in S \\ \mathcal{M}_1 & \text{otherwise} \end{cases}$$

We write $n \in S_s$ if $R(n,s)$ holds. In particular, $n \in S$ if and only if there are infinitely many stages s such that $n \in S_s$. We begin, as before, with $B = \omega$, and with $f_{-1} = D_{-1} = \emptyset$.

At stage s we will adjust the function f to conform to our changing beliefs about whether $n \in S$. If we have $n \notin S_s$, then we extend f_{s-1} so that $dom(f_s)$ includes the first s elements of B and $ran(f_s)$ includes the first s elements of $\mathcal{M}_1$.

On the other hand, if we have $n \in S_s$, then we want to collapse the dimension of the model we are building. Let t be the latest stage before s such that either $t = 0$ or $n \notin S_t$. Now f_{s-1} maps some elements $\bar{d}$ of B into $P(\emptyset)_s$, and some other elements $c_1, \ldots, c_j$ to $P(a_0)_s - P(\emptyset)_s$. Let $\delta_s(\bar{d}, c_1, \ldots, c_j)$ be the conjunction of all elements of D_{s-1}. Now $\mathcal{M}_1 \models \exists \bar{x}\ \delta_s(\bar{d}, \bar{x})$, and so we also have $\mathcal{M}_0 \models \exists \bar{x}\ \delta_s(\bar{d}, \bar{x})$. Let $\tilde{c}_1, \ldots, \tilde{c}_j$ be such that $\mathcal{M}_0 \models \delta_s(\bar{d}, \tilde{c}_1, \ldots, \tilde{c}_j)$. We can now adjust the map f so that

$$f_s(x) = \begin{cases} \tilde{c}_i & \text{if } x = c_i \\ f_t(x) & \text{otherwise} \end{cases}$$

so that f_s extends f_t.

In any case, we also find the first atomic sentence λ such that neither λ nor its negation is in D_{s-1}, but all constants appearing in λ are in $dom(f_s)$. We compute its truth value via f_s, and add the result to D_{s-1} to form D_s.

A true stage s is one in which if $n \in S$ then $n \in S_s$ and if $n \notin S$ then for any $t \geq s$ we have $n \notin S_t$. Note that if $s \leq s'$ are true stages, then $f_s \subseteq f_{s'}$. We define

T to be the set of all true stages, and $f = \bigcup_{s \in T} f_s$, and let $\mathcal{A}_n$ be the structure with universe B whose atomic diagram is given by $\bigcup_{s \in \omega} D_s$.

Now if $n \in S$, we will infinitely often see $n \in S_s$, so that $\mathcal{A}_n \simeq_f \mathcal{M}_0$. On the other hand, if $n \notin S$, then after some stage f_s will always be copying $\mathcal{M}_1$ and $f : \mathcal{A}_n \simeq \mathcal{M}_1$. □

Theorem 3.4.5. *Let T be a strongly minimal theory with effective elimination of quantifiers and a computable model, where $acl(\emptyset)$ is infinite, and let $\mathcal{M}$ be a computable model of T which has finite dimension $m > 0$. If K_f is the class of finite dimensional models of T, then $I(\mathcal{M})$ is m-complete d-Σ_2^0 within K_f.*

Proof. Let $\mathcal{M}_{m-1} \prec \mathcal{M}_m \prec \mathcal{M}_{m+1}$, where $\mathcal{M}_k$ has dimension k. Again, the bound comes from the methods of Theorem 3.4.1. The copies of $\mathcal{M}$ are those models of T which satisfy the sentence $D_m \wedge \neg D_{m+1}$.

Toward completeness, let $S = S_1 - S_2$, where $S_i = \exists^{<\infty} x\ R_i(n, x)$ are arbitrary Σ_2^0 sets. We will construct a uniformly computable sequence of models $(\mathcal{A}_n)_{n \in \omega}$ so that

$$\mathcal{A}_n \simeq \begin{cases} \mathcal{M}_m & \text{if } n \in S \\ \mathcal{M}_{m-1} & \text{if } n \notin S_1 \\ \mathcal{M}_{m+1} & \text{if } n \in S_1 \cap S_2 \end{cases}$$

We begin, as usual, with $B = \omega$ and with $f_{-1} = D_{-1} = \emptyset$. We also set $q_{-1} = m - 1$.

We say that $n \in S_{i,s}$ if and only if we have $\neg R_i(n, s)$. In particular, $n \in S_i$ if and only if for sufficiently large s we have $n \in S_{i,s}$.

At stage s, we will first define the target dimension d_s. We set

$$d_s = \begin{cases} m & \text{if } n \in S_{1,s} - S_{2,s} \\ m - 1 & \text{if } n \notin S_{1,s} \\ m + 1 & \text{if } n \in S_{1,s} \cap S_{2,s} \end{cases}.$$

If $d_{s-1} > d_s$, then we will collapse the dimension back to d_s. Let t be the last stage before s such that $d_t = d_s$ and such that for all z between t and s, we have $d_z \geq d_t$. Now f_{s-1} maps some elements $\bar{d}$ of B into $P(\bar{a}_{\{i|i<d_s-1\}})_s$, and some other elements $c_1, \ldots, c_j$ to $P(\bar{a}_{\{i|i<m+1\}})_s - P(\bar{a}_{\{i|i<d_s-1\}})_s$. Let $\delta_s(\bar{d}, c_1, \ldots, c_j)$ be the conjunction of all elements of D_{s-1}. Now $\mathcal{M}_{m+1} \models \exists \bar{x}\ \delta_s(\bar{d}, \bar{x})$, and so we also have $\mathcal{M}_{d_s} \models \exists \bar{x}\ \delta_s(\bar{d}, \bar{x})$. Let $\tilde{c}_1, \ldots, \tilde{c}_j$ be such that $\mathcal{M}_{m-1} \models \delta_s(\bar{d}, \tilde{c}_1, \ldots, \tilde{c}_j)$. We can now adjust the map f so that

$$f_s(x) = \begin{cases} \tilde{c}_i & \text{if } x = c_i \\ f_t(x) & \text{otherwise} \end{cases}$$

and so that $dom(f_s)$ contains the first s elements of B and $ran(f_s)$ contains the first s elements of $\mathcal{M}_{d_s}$. Thus, we will have $f_s \supseteq f_t$. Also, we find the first atomic sentence λ such that neither λ nor its negation is in D_{s-1}, but all constants appearing in λ are in $dom(f_s)$. We compute its truth value via f_s, and add the result to D_{s-1} to form D_s.

We say that a stage s is a *true stage* if and only if for all $t > s$ we will never have $d_t < d_s$.

Lemma 3.4.6. *There are infinitely many true stages.*

Proof. Suppose $n \in S_1 \cap S_2$. Then there is some t_0 such that for $t > t_0$ we will have $n \in S_{1,t} \cap S_{2,t}$, so that $d_t = m + 1$. Consequently, every $t > t_0$ is a true stage.

If $n \notin S_1$, then for infinitely many s we will have $n \notin S_{1,s}$, so that for all such s we will have $d_s = m - 1$. Since $d_t \geq m - 1$ for all t, these s are true stages.

Finally, if $n \in S$, then there is some t_1 such that for $t > t_1$ we have $n \in S_{1,t}$, but there are infinitely many s such that $n \notin S_{2,s}$. Take t such that $n \notin S_{2,t}$ but $t > t_1$. Now $d_t = m$, and for $s > t$ we will always have either $d_s = m$ or $d_s = m+1$, so that t is a true stage. ∎

Let $\mathcal{A}_n$ be the structure with universe B and with atomic diagram $\bigcup\limits_{s\in\omega} D_s$. It is isomorphic to a substructure of $\mathcal{M}_{m+1}$ via $f = \bigcup\limits_{s\in T} f_s$, where T is the set of all true stages. It is clear that $dim(\mathcal{A}_n)$ will be what was claimed. □

3.5 The Isomorphism Problem for Computable Archimedean Real Closed Fields

Since the proof of Theorem 3.3.1 used the non-Archimedean elements in a central way, it is worthwhile to consider what happens when this possibility is removed. In fact, things become much simpler.

Theorem 3.5.1. *If K is the class of Archimedean real closed fields, then $E(K)$ is Π^0_3 complete within K.*

Proof. The class of real closed fields can be axiomatized by a computable infinitary Π_2 sentence, as can the class of Archimedean real closed fields (by adding the sentence that for each element x, some finite multiple of 1 is greater than x). Archimedean real closed fields are classified simply by the cuts that are filled, so the statement

$$\forall x\ \exists \hat{x} \bigwedge\!\!\!\bigwedge_{q\in\mathbb{Q}} [(\mathcal{A}_a \models q \leq x \Leftrightarrow \mathcal{A}_b \models q \leq \hat{x})] \wedge$$

$$\forall z\ \exists \hat{z} \bigwedge\!\!\!\bigwedge_{q\in\mathbb{Q}} [(\mathcal{A}_b \models q \leq z \Leftrightarrow \mathcal{A}_a \models q \leq \hat{z})]$$

defines the relation $\mathcal{A}_a \simeq \mathcal{A}_b$, showing that it is, at worst, Π^0_3.

Lemma 3.5.2. *There exists a uniformly computable sequence $(a_i)_{i\in\omega}$ of real numbers which are algebraically independent.*

Proof. Lindemann's theorem states (in one form) that if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are algebraic numbers linearly independent over the rationals, then $e^{\lambda_1}, \ldots, e^{\lambda_k}$ are algebraically independent [7]. Further, it is well known that the set $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots\}$

is linearly independent (a proof may be found in [11]). Alternately, we could bypass this technology and simply build the sequence by a priority argument. □

Now consider the language $(+, \cdot, 0, 1, (a_i)_{i\in\omega})$, and the theory consisting of the axioms of real closed fields and the further axioms $q < a_i$ and $q > a_i$. This is a complete decidable theory, and so has a computable model $\mathcal{A}^*$. The set of elements of $\mathcal{A}^*$ algebraic over the set of a_i is c.e. and by padding we can find a computable structure $\mathcal{M} = (M, +, \cdot, 0, 1, \leq, (a_i)_{i\in\omega})$ where M is the real closure of $(a_i)_{i\in\omega}$.

Toward the completeness part of the theorem, we will produce a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that $\mathcal{A}_n \simeq \mathcal{A}^*$ if and only if $n \notin cof$. Let $D_{-1} = T_{-1} = f_0 = \emptyset$, and let B be an infinite computable set. If $C \subseteq \omega$, then we write RCF^C for the theory of real closed fields with constants for $\{a_i | i < |C|\}$, and P_C the computable prime model of RCF_C within $\mathcal{A}^*$.

We begin by defining our estimates for $S = \omega - W_n$. Let $S_0 = \emptyset$. At stage s, if $S_{s-1} \subseteq \omega - W_{n,s-1}$, then we find the furst element x outside $W_{n,s}$ and let $S_s = S_{s-1} \cup \{x\}$. On the other hand, if $S_{s-1} \nsubseteq \omega - W_{n,s-1}$, then we find the least $r \in S_s \cap W_{n,s}$ and set $S_s = S_{s-1} \cap \{q | q < r\}$. Now to build $\mathcal{A}_n$, we will proceed as follows.

At each stage s where $S_{s-1} \subseteq S_s$, extend f_{s-1} to f_s so that its domain includes the first s members of B and so that its range includes the first s members of P_{S_s}.

At each stage s where $S_{s-1} \nsubseteq S_s$, we will have to collapse to a smaller model. There are some elements in $ran(f_{s-1})$ which should not be in $ran(f)$. Let t be the last stage less than s such that $S_t \subseteq S_s$. Now f_{s-1} maps some elements $\bar{d}$ of B into $P_{S_t,s}$, and some other elements $c_1, \ldots, c_j$ to $P_{S_s,s} - P_{S_t,s}$. Let $\delta_s(\bar{d}, c_1, \ldots, c_j)$ be the conjunction of all elements of D_{s-1}. Now $P_{S_s} \models \exists \bar{x}\ \delta_s(\bar{d}, \bar{x})$, and so we also have $P_{S_t} \models \exists \bar{x}\ \delta_s(\bar{d}, \bar{x})$. We run the enumeration until we see the witnesses $\tilde{c}_1, \ldots, \tilde{c}_j$ such that $P_{S_t} \models \delta_s(\bar{d}, \tilde{c}_1, \ldots, \tilde{c}_j)$. We now define f_s so that $f_s \supseteq f_t$, so that $f_s(c_i) = \tilde{c}_i$, and so that $dom(f_s)$ includes the first s elements of B and $ran(f_s)$ includes the first s elements of P_{S_s}.

Otherwise no such collpapse is necessary. We then simply extend f_{s-1} to f_s so that $dom(f_s)$ includes the first s elements of B and $ran(f_s)$ includes the first s elements of P_{S_s}.

In any case, we also extend D_{s-1} to D_s as follows. We find the first atomic sentence λ such that neither λ nor $\neg\lambda$ is in D_{s-1} but all constants used in λ are in $dom(f_s)$, compute its truth value via f_s, and record the result in D_s.

A *true stage* is a stage s such that $S_s \subseteq \omega - W_n$.

Lemma 3.5.3. *There are infinitely many true stages.*

Proof. Consider some non-true stage s. Suppose that r^* is the least element of S_s which is not in $\omega - W_n$. There is some least stage $t \geq s$ at which $r^* \in W_{n,t}$. Now $t+1$ will be a true stage. ∎

Let $\mathcal{A}_n$ be the structure with atomic diagram $\bigcup\limits_{s\in\omega} D_s$, and universe B. Let $f = \bigcup\limits_{s\in T} f_s$, where T is the set of all true stages. Now $\mathcal{A}_n$ is isomorphic to a substructure of $\mathcal{A}^*$ via f.

Now $\mathcal{A}_n$ is isomorphic to the substructure of $\mathcal{A}^*$ which is the real closure of $\bar{a}_{\bar{W}_n}$. Thus, $\mathcal{A}_n$ has elements filling the same cuts as the first $|\bar{W}_n|$ elements of $\{a_i | i \in \omega\}$. Consequently, $\mathcal{A}_n \simeq \mathcal{A}^*$ if and only if W_n is co-infinite. ∎

3.6 Index Sets for Computable Archimedean Real Closed Fields

For previous classes of fields, it seems out of reach to completely describe the complexity of the index sets of all possible models. On the other hand, we have an idea of the structure of Archimedean real closed fields which is sufficient to give such a treatment.

The proof of Theorem 3.5.1 suggests that infinite transcendence degree is an important aspect of the index set having maximal complexity. It remains to ask whether *any* Archimedean real closed field with infinite transcendence degree behaves similarly, or whether the result is only true in some special fields that have sequences like those guaranteed in Lemma 3.5.2. The results of the present section are preliminary work of the present author with Harizanov, Knight, and S. Miller which will be more fully reported in [16]

As it turns out, given an Archimedean ordered field $\mathcal{A}$ of infinite transcendence degree and a Π^0_3 set S, we can construct a uniformly computable sequence of Archimedean real closed fields $(\mathcal{A}_n)_{n\in\omega}$ such that $\mathcal{A}_n \simeq \mathcal{A}$ if and only if $n \in S$ by treating all cuts alike — real algebraic or not. Some will necessarily get filled and others will be filled or not according to whether $n \in \omega$, but we do not need to know which is which.

Theorem 3.6.1. *Let K be the class of computable Archimedean ordered fields, and let $\mathcal{A}$ be a member of K. Assume $\mathcal{A}_0$ is a purely transcendental extension of $\mathbb{Q}$ and that $\mathcal{A}$ is either a finite extension of $\mathcal{A}_0$ or its real closure.*

1. *If the transcendence degree of $\mathcal{A}$ is 0, we have that $I(\mathcal{A})$ is m-complete Π^0_2 within K.*
2. *If the transcendence degree of $\mathcal{A}$ is finite but greater than 0, we have that $I(\mathcal{A})$ is m-complete d-Σ^0_2 within K.*
3. *If the transcendence degree of $\mathcal{A}$ is infinite we have that $I(\mathcal{A})$ is m-complete Π^0_3 within K.*

Question 3.6.2. *Let $\mathcal{A}$ be a computable Archimedean ordered field of infinite transcendence degree. Is there a uniformly computable sequence of algebraically independent elements of $\mathcal{A}$?*

Question 3.6.3. *Suppose that $\mathcal{A}$ is an Archimedean ordered field which is not real closed and is an infinite algebraic extension of $\mathcal{A}_0$, where $\mathcal{A}_0$ is as in Theorem 3.6.1. What is the m-degree of $I(\mathcal{A})$?*

CHAPTER 4

FINITE STRUCTURES AND COMPARISONS

So far our attention has been restricted to infinite structures, and this is not without reason. Since it is difficult to find anything about a finite structure that is very far from being computable, it at first appears that computability theory is too blunt a tool for any meaningful analysis of such structures.

In the first section of this chapter, we will give evidence in favor of this obvious prejudice. Afterwards, though, we will see a method first proposed in joint work of the present author with Cummins, Knight, and S. Miller [15], by which meaningful and non-trivial comparisons can be made between *classes* of finite structures. Moreover, the method is also interesting when applied to classes of infinite structures, and will find important application in Chapter 5 for calculating Scott ranks.

4.1 Index Sets of Finite Structures

It is well-known that any finite structure is determined up to isomorphism by a relatively simple first-order sentence. The next theorem could be considered an optimality result for that sentence. On the other hand, it is an analogue of the index set results of Section 3.4, with finite sizes corresponding to finite dimensions. In that sense, this result relates results like those of Section 3.4 to their "lower end."

Theorem 4.1.1 (C.–Harizanov–Knight–S. Miller [16]). *Let $\mathcal{A}$ be a finite structure for a language $\mathcal{L}$, and let K be the class of finite structures for that language.*

1. *If $\mathcal{A}$ is empty, then $I(\mathcal{A})$ is m-complete Π^0_1 within K.*

2. *If $\mathcal{A}$ has size ≥ 1, then $I(\mathcal{A})$ is m-complete d-c.e. within K.*

4.2 Comparing Classes of Structures

While the *internal* complexity of a single finite structure is rather severely bounded, the *external* complexity — its situation within a class of similar structures — can still be quite complex. No one would say, for instance, that the class of finite graphs is "simple." In an earlier chapter, allusion was made to a method of comparing classes of structures, which originated in a paper of Friedman and Stanley.

Definition 4.2.1 (Friedman–Stanley [25]). *View the set of structures in a fixed language and with universe ω as a topological space via the product topology. Let K_1 and K_2 be classes of countable structures with the subspace topology. Then we say $K_1 \leq_B K_2$ if there is a Borel measurable function $\Phi : K_1 \to K_2$ which is well-defined and 1-1 on isomorphism types.*

Clearly if the classes of structures involved are countable, this definition is completely useless, since any function between countable sets is Borel measurable. This fails to account, though, for the unreasonable effectiveness of proofs that $K_1 \leq_B K_2$ in proving theorems about isomorphism problems for classes of computable structures, such as Theorems 3.2.4 and 3.3.1.

In an effort to explain this, the present author, Cummins, Knight, and S. Miller proposed a definition of a different ordering, analogous to $\leq_B$ but based on enumeration reducibility which does give some information about these classes. This definition is by no means the only reasonable analogy. However, it is a definition that fairly reflects the situation of finite structures, and about which we can prove interesting results.

Definition 4.2.2 (C.–Cummins–Knight–S. Miller [15]). *Let K_1 and K_2 be classes of structures, and let Φ be a c.e. set of pairs (α, φ), where α is a subset of the atomic diagram of a finite structure for the language of K_1, and φ is an atomic sentence, or the negation of one, in the language of K_2.*

1. *We say that Φ is a* computable transformation from K_1 to K_2 *if for all $\mathcal{A} \in K_1$, the set $\Phi(D(\mathcal{A}))$ has the form $D(\mathcal{B})$, for some $\mathcal{B} \in K_2$. We may write $\Phi(\mathcal{A}) = \mathcal{B}$ (identifying the structures with their atomic diagrams).*

2. *We say that $K_1 \leq_c K_2$ if there is a computable transformation Φ from K_1 to K_2, which, when viewed as a function $\Phi : K_1 \to K_2$, is well-defined and 1-1 on isomorphism types.*

The encoding used in the proof of Theorem 3.2.4 is an example of a computable transformation witnessing that if UG is the class of undirected graphs and F is the class of fields, then $UG \leq_c F$. Cummins used an argument from Friedman and Stanley [25] that if LO is the class of linear orders and DG is the class of directed graphs then $DG \leq_B LO$ to show that $DG \leq_c LO$ [20]. An important general property of computable transformations is the following proposition.

Proposition 4.2.3 (C.–Cummins–Knight–S. Miller [15]). *Let K_1 and K_2 be classes of structures, and let Φ be a computable transformation from K_1 to K_2. If $\mathcal{A}, \mathcal{A}' \in K_1$, where $\mathcal{A} \subseteq \mathcal{A}'$, then $\Phi(\mathcal{A}) \subseteq \Phi(\mathcal{A}')$.*

This apparently minor fact turns out to be very important in proving results of the form $K_1 \not\leq_c K_2$, which is the harder part of almost any comparison of classes using $\leq_c$. For the remainder of this section we will adopt the following operational definition.

Definition 4.2.4. *We will use the following names for some important classes of structures.*

$$\begin{array}{lcl} G & = & \textit{Graphs} \\ VS & = & \textit{Vector Spaces} \\ FVS & = & \textit{Finite-dimensional Vector Spaces} \\ FG & = & \textit{Finite Graphs} \\ CG & = & \textit{Cyclic graphs} \end{array}$$

4.2.1 Basic Properties of the Order $\leq_c$

The natural classes just named are conveniently situated as landmarks throughout the partial order of all classes of countable structures under $\leq_c$.

Theorem 4.2.5 (C.–Cummins–Knight–S. Miller [15]). *Among the classes listed above, we have the following relations:*

$$CG <_c FG <_c FVS <_c VS <_c G$$

We say that two classes K_1 and K_2 are c-equivalent (written $K_1 \equiv_c K_2$, and often just expressed by "equivalent" if no confusion is likely) if both $K_1 \leq_c K_2$ and $K_2 \leq_c K_1$. It turns out that most natural classes of structures are equivalent to one of these classes.

Theorem 4.2.6 (C.–Cummins–Knight–S. Miller [15]). *We have the following equivalences*

1. *The following classes are c-equivalent:*
   - (a) *CG*
   - (b) *Finite prime fields*
2. *The following classes are c-equivalent:*
   - (a) *Finite graphs*
   - (b) *Finite groups*
   - (c) *Finite simple groups*

(d) Finite cyclic groups

(e) Finite linear orders

(f) Arbitrary finite structures

As might be expected, all classes of finite structures fall below FG and all classes of countable structures fall below G. While most natural classes of structures tend to fall into the linearly ordered segment described in Theorem 4.2.5, there are many other equivalence classes. One important method of constructing them is based on immunity.

Definition 4.2.7 (C.–Cummins–Knight–S. Miller [15]). *Let $X \subseteq \omega$. We say that sets $A, B \subseteq \omega$ are X* bi-immune *if for any X-computable function f with infinite range, there is some $a \in A$ such that $f(a) \notin B$, and there is some $b \in B$ such that $f(b) \notin A$. We say that A and B are* bi-immune *if they are X bi-immune for computable X.*

If A and B are bi-immune, and if K_A is the class of cyclic graphs of size n for $n \in \mathcal{A}$ and K_B for $n \in B$, then neither $K_A \leq_c K_B$ nor $K_B \leq_c K_A$.

Proposition 4.2.8 (C.–Cummins–Knight–S. Miller [15]). *For any set X, there exists a family $(A_f)_{f \in 2^\omega}$ such that for any distinct $f, g \in 2^\omega$, A_f and A_g are X bi-immune.*

Thus, there are continuum many classes of computable structures which are $\leq_c$ incomparable, and which are $\leq_c$-below the class of finite cyclic graphs. Many similar results are possible, and the partial ordering given by $\leq_c$ is quite complicated.

Theorem 4.2.9 (Knight [44]). *There are classes K_1 and K_2 of structures such that K_1 and K_2 have neither least upper bound nor greatest upper bound. Thus, $\leq_c$ is not a lattice.*

4.2.2 Applications to Infinite Structures

The ordering $\leq_c$ was originally developed to investigate classes of finite structures. However, it gives some important information, too, about infinite structures. A first taste of this is a byproduct of the immunity investigations.

An important question about isomorphism problems is exactly which m-degrees can occur as the degree of the isomorphism problem for some class of computable structures. It is generally believed that far more degrees are possible than we have actually seen. A major milestone, though, is producing two such degrees that are incomparable.

Theorem 4.2.10. *There are incomparable m-degrees each of which is the degree of the isomorphism problem for some class of computable structures.*

Proof. Let A and B be Δ^0_4 bi-immune sets of natural numbers. Let K_A be the set of computable $\mathbb{Q}$-vector spaces of dimensions from A, and let K_B be defined symmetrically. Then $E(K_A)$ and $E(K_B)$ are incomparable. □

More applications that come directly from computable transformations and the relation $\leq_c$ are being seen in work currently in progress. Much of the effectiveness of these concepts comes from the following theorem of Knight.

Theorem 4.2.11 (Knight [44]). *Let Φ witness that $K_1 \leq_c K_2$. For any computable infinitary sentence φ in the language of K_2, we can find a computable infinitary sentence φ^* in the language of K_1 such that for all $\mathcal{A} \in K_1$ we have $\Phi(\mathcal{A}) \models \varphi$ if and only if $\mathcal{A} \models \varphi^*$. Moreover, if φ is of class Σ^0_α, then φ^* may also be chosen to be of class Σ^0_α.*

CHAPTER 5

COMPUTABLE STRUCTURES AND HIGH SCOTT RANK

We have already seen examples of highly complicated structures. For instance, the way in which a class of computable structures with Σ^1_1 complete isomorphism problem are proved to be at that level is typically that it has some member whose index set is of the greatest possible complexity.

With the exception of the background information in Section 5.1, all of the material in the present chapter is of a preliminary nature. This chapter is meant to give the reader a sense of the state of a very important and promising direction for future inquiry in the subject of this thesis, however incomplete the work on this direction may be.

5.1 Scott Rank

Question 5.1.1. *What gives a structure high index set complexity?*

One measure of the internal complexity of a structure is the Scott rank. This rank is one good candidate for an answer to the question. There are several reasonable definitions of the Scott rank of a structure in the literature, and all are nearly equivalent (most differ only finitely, and many agree on limit ordinals; see [5] for more details). The following definition allows for reasonable calculations, and makes fine distinctions at the top of the range. We first define the Scott rank of a tuple of elements.

Definition 5.1.2.

1. *$\bar{a} \sim_0 \bar{b}$ if they satisfy all the same atomic formulas.*

2. *$\bar{a} \sim_{\alpha+1} \bar{b}$ if for every d there is a c such that $\bar{b}d \sim_\alpha \bar{a}c$.*

3. *$\bar{a} \sim_\gamma \bar{b}$ for limit γ if $\bar{a} \sim_\alpha \bar{b}$ for all $\alpha < \gamma$.*

4. *$sr(\bar{a})$ is the least ordinal α such that $\bar{b} \sim_\alpha$ implies $\bar{b} \simeq \bar{a}$.*

With this definition in hand, we proceed to define the Scott rank of a structure.

Definition 5.1.3. *Given a structure $\mathcal{A}$, we define the Scott rank of $\mathcal{A}$, denoted $SR(\mathcal{A})$ to be the least ordinal greater than $sr(\bar{a})$ for all finite $\bar{a} \subseteq \mathcal{A}$.*

Nadel [55] showed that if $\mathcal{A}$ is a computable structure, then $SR(\mathcal{A}) \leq \omega_1^{CK} + 1$. A computable linear ordering of type $\omega_1^{CK}(1 + \eta)$ (Harrison showed that such an ordering exists [33]), where η is the order type of the rationals, has Scott rank $\omega_1^{CK} + 1$, since, for instance, the orbit of the first element in a copy of ω_1^{CK} other than the first, is not hyperarithmetical. If $\mathcal{A}$ is a computable infinite dimensional $\mathbb{Q}$-vector space, then $SR(\mathcal{A}) = 1$.

It will be helpful to have on hand a few additional easy properties of Scott rank:

Theorem 5.1.4 (Classical). *Let $\mathcal{A}$ be a computable structure.*

1. *We have $SR(\mathcal{A}) < \omega_1^{CK}$ if and only if the structure $\mathcal{A}$ has a computable infinitary Scott sentence.*

2. *We have $SR(\mathcal{A}) = \omega_1^{CK} + 1$ if and only if there is some element $x \in \mathcal{A}$ such that $sr(x) = \omega_1^{CK}$.*

The possibility of a relationship of Scott rank to index set complexity is expressed in a question of Goncharov and Knight.

Question 5.1.5 (Goncharov–Knight [31]). *If $I(\mathcal{A})$ is hyperarithmetical, must $SR(\mathcal{A})$ be computable?*

It is not at all clear what the answer to this question will be. Much of the work discussed in this chapter arises from attempts to build intuition and techniques in the general vicinity of this problem, but we are not yet prepared to mount a full frontal assault on it.

Rather, this chapter will document joint work of the present author with several others, first on the construction of various examples of structures with Scott rank ω_1^{CK} exactly, and then on the complexity of the set of indices for all structures with non-computable Scott rank.

5.2 Examples of Structures of High Scott Rank

As has already been noted, there are many examples of computable structures $\mathcal{A}$ where either $SR(\mathcal{A})$ is computable or $SR(\mathcal{A}) = \omega_1^{CK} + 1$. There are no examples in which $SR(\mathcal{A}) > \omega_1^{CK} + 1$. It is natural, then, to look at the gap:

Question 5.2.1. *Is there a computable structure $\mathcal{A}$ where $SR(\mathcal{A}) = \omega_1^{CK}$?*

It is important to note that some other definitions of Scott rank collapse ranks ω_1^{CK} and $\omega_1^{CK} + 1$. Nadel's paper [55], for instance, calculates the maximum Scott rank (in the sense of the definition of Barwise [10]) of a computable structure to be ω_1^{CK}, so that the problem is trivial. In our definition, though, the distinction exists.

It at first seemed that structures of Scott rank ω_1^{CK} should at least be rare and unnatural. The earliest examples were.

Theorem 5.2.2 (Makkai [48]). *There is an arithmetical structure $\mathcal{A}$ such that $SR(\mathcal{A}) = \omega_1^{CK}$.*

Theorem 5.2.3 (Knight–Young [45]). *There is a computable structure $\mathcal{A}$ such that $SR(\mathcal{A}) = \omega_1^{CK}$.*

These structures were not of any type to which we are accustomed, which presented difficulties in understanding them. To produce these structures, one would start with a tree, define certain group structures in the level of the tree, and use the result to produce a new structure which included neither the original tree nor any of the group operations. The purpose of introducing the group structure is to impose a property of "homogeneity" on the structure (this is not the usual model-theoretic notion of homogeneity), so that tuples that appear to be automorphic actually are automorphic.

The present author, along with Knight and Young, noticed that this homogeneity could be imposed from the beginning, allowing an example to be given which is simply a tree.

Definition 5.2.4 (C.–Knight–Young [18]). *We say that a tree T is* rank homogeneous *if for every $n \in \omega$ and every $x \in T$ at level n, the following are satisfied:*

1. *For every $\alpha < tr(x)$, if there is $y \in T$ at level $n+1$ such that $tr(y) = \alpha$, then there is an infinite set of nodes $\{y_i\}_{i\in\omega} \subseteq T$, each of which is at level $n+1$, each of which is a successor of x, and each of which has rank α.*

2. *If $tr(x) = \infty$, then x has infinitely many successors of rank ∞.*

The most important thing about this class of trees is that if T is a rank homogeneous tree, then we can calculate the Scott ranks of elements in T. In particular, if $\bar{x}$ is a finite subset of T and for each $x \in \bar{x}$ we have $tr(x)$ computable, then $sr(\bar{x})$ is computable. Further, if $x, y \in T$ at level n, and $tr(x) = tr(y) = \infty$, then x and y are automorphic. It follows that if for each level n there is a computable ordinal α_n bounding the tree ranks of elements x at level n which have $tr(x) < \infty$, then all tuples in T have computable Scott rank. On the other hand, if for any computable ordinal α there exists $x \in T$ with $\alpha < tr(x) < \infty$, then there is no computable bound on the Scott ranks of tuples, so that $SR(T) \geq \omega_1^{CK}$. We can now summarize the additional property we need in our tree.

Definition 5.2.5. *We say that a tree T is* thin *if for every level n there is a computable ordinal α_n bounding the nodes at level n with ordinal rank, but for any computable ordinal α there exists $x \in T$ with $\alpha < tr(x) < \infty$.*

What is needed, then, is a thin, rank homogeneous tree. We can do exactly that.

Theorem 5.2.6 (Knight–Young [45]). *There is a computable thin tree.*

Theorem 5.2.7 (C.–Knight–Young [18]). *From an index for a computable tree T we can pass effectively to an index for a computable rank homogeneous tree $\tilde{T}$ such that the ranks that occur at level n in $\tilde{T}$ are exactly the ones that occur at level n in T.*

Theorem 5.2.8 (C.–Knight–Young [18]). *If T is a computable thin rank homogeneous tree, then $SR(T) = \omega_1^{CK}$.*

Since we have a tree of Scott rank ω_1^{CK}, it becomes believable that there are examples in other interesting classes of structures. Work in progress, summarized by the following two theorems, gives such examples.

Theorem 5.2.9 (C.–Goncharov–Knight). *Let K be any of the following classes. Then there is a computable structure $\mathcal{A} \in K$ such that $SR(\mathcal{A}) = \omega_1^{CK}$.*

1. *Undirected graphs*

2. *Linear orders*

3. *Fields of characteristic 0*

It appears, then, that structures of rank ω_1^{CK} are not as exotic as was once believed. We might reasonably expect to encounter them in more classes, or in problems in which we did not set out to find them. On the other hand, there are some contexts in which we can exclude the possibility of such a structure.

Theorem 5.2.10 (C.–Goncharov–Knight). *Let K be any of the following classes. Then if $\mathcal{A}$ is a computable member of K and $SR(\mathcal{A}) \geq \omega_1^{CK}$, then $SR(\mathcal{A}) = \omega_1^{CK} + 1$.*

1. *Abelian p-groups*
2. *Models of the computable infinitary theory of well-orderings*
3. *Models of the computable infinitary theory of superatomic Boolean algebras*

5.3 Index Sets for High Scott Rank

In examples we have explored in earlier sections of this thesis, a calculation of the complexity of the index set of a structure requires some deep understanding of the structure itself, and in turn gives some insight on what sort of information the structure contains. It is reasonable to think that this would happen not only for structures, but also for classes of structures — for instance, the class K_{HR} of computable structures of noncomputable Scott rank. We will also examine $I(K_\alpha)$ where K_α is the class of computable structures of Scott rank α and where $\alpha \in \{\omega_1^{CK}, \omega_1^{CK} + 1\}$. In this section, we will see the results of work in progress, which has an eventual goal of resolving the following question, which is currently open.

Question 5.3.1. *What are the m-degrees of the following?*

1. $I(K_{HR})$
2. $I(K_{\omega_1^{CK}})$
3. $I(K_{\omega_1^{CK}+1})$

Naive ways to write the definition $L_{\omega_1\omega}$ with second order quantifiers give the following bounds.

Proposition 5.3.2 (C.–Goncharov–Knight–Kudinov–Morozov–Puzarenko).

1. *$I(K_{HR})$ is Σ^1_1.*

2. *$I(K_{\omega_1^{CK}})$ has a definition of the form $\forall\exists(d\text{-}\Sigma^1_1)$.*

3. *$I(K_{\omega_1^{CK}+1})$ has a definition of the form $\exists\forall(\Sigma^1_1 \vee \Pi^1_1)$.*

Of course, for the first item, we have much more exact information, since if K is the class of orderings isomorphic to either a computable ordinal or to $\omega_1^{CK}(1+\eta)$, then $I(K \cap K_{HR})$ is m-complete Σ^1_1 within K.

Proposition 5.3.3 (C.–Goncharov–Knight–Kudinov–Morozov–Puzarenko). *$I(K_{HR})$ is m-complete Σ^1_1.*

Completeness in the other two cases remains open. If S is a Π^1_1 set, we can show that $S \leq_m I(K_{\omega_1^{CK}+1})$ within K_{HR} using a method of "hiding" a copy of the Harrison ordering. We will construct a uniformly computable sequence of structures $(\mathcal{A}_n)_{n\in\omega}$ such that $\mathcal{A}_n$ is of Scott rank ω_1^{CK} We begin with a Harrison ordering, and to each pair of elements we connect an infinite set. For each n, we will give each of these sets the structure of a tree.

Lemma 5.3.4. *Let $S \in \Pi^1_1$, and let T be the computable tree of Scott rank ω_1^{CK} constructed in Theorems 5.2.6, 5.2.7, and 5.2.8. There is a uniformly computable sequence of trees $(\mathcal{C}_n)_{n\in\omega}$ such that $\mathcal{C}_n \simeq T$ if and only if $n \notin S$, and such that if $n \in S$ we have $SR(\mathcal{C}_n) < \omega_1^{CK}$.*

Now if (a, b) is a pair in the ordering, the structure attached to the pair (a, b) will be the structure $\mathcal{C}_n$ if $a < b$ and T otherwise. The structure $\mathcal{A}_n$ will be the reduct in which the original ordering is forgotten and we retain only the structures attached to the various pairs. In the case that $n \in \S$, the structure $\mathcal{C}_n$ will have a computable infinitary Scott sentence, and so we can recover the Harrison ordering, giving some tuples Scott rank ω_1^{CK} and making $SR(\mathcal{A}_n) = \omega_1^{CK} + 1$. On the other hand, if $n \notin S$, we will have $SR(\mathcal{A}_n) = \omega_1^{CK}$. Lemma 5.3.4 was a major motivation for proving Theorems 5.2.7 and 5.2.8, since it was not clear that an analogous result would hold for previous examples of structures with Scott rank ω_1^{CK}.

The method just described, along with the fact that if K is the class of orderings isomorphic to either a computable ordinal or to $\omega_1^{CK}(1+\eta)$, then $I(K \cap K_{HR})$ is m-complete Σ_1^1 within K, are the principal methods needed to get the strongest known completeness results. We state these results here.

Proposition 5.3.5 (C.–Goncharov–Knight–Kudinov–Morozov–Puzarenko). *We have the following bounds:*

1. *For any set S with a definition of the form $\exists(d\text{-}\Sigma_1^1)$, we have $S \leq_m I(K_{\omega_1^{CK}+1})$.*

2. *For any set S with a definition of the form $\forall(\Sigma_1^1 \vee \Pi_1^1)$, we have $S \leq_m I(K_{\omega_1^{CK}+1})$.*

BIBLIOGRAPHY

[1] S. Abhyankar. On the compositum of algebraically closed subfields. *Proceedings of the American Mathematical Society*, 7:905–907, 1956.

[2] C. J. Ash. Recursive labelling systems and stability of recursive structures in hyperarithmetical degrees. *Transactions of the American Mathematical Society*, 298:497–514, 1986. Corrections: *Ibid*, vol. 300 (1988) p. 851.

[3] C. J. Ash. Labelling systems and r.e. structures. *Annals of Pure and Applied Logic*, 47:99–119, 1990.

[4] C. J. Ash. Generalizations of enumeration reducibility using recursive infinitary propositional sentences. *Annals of Pure and Applied Logic*, 58:173–184, 1992.

[5] C. J. Ash and J. F. Knight. *Computable structures and the hyperarithmetical hierarchy*. Elsevier, 2000.

[6] R. Baer. Abelian groups without elements of finite order. *Duke Mathematical Journal*, 3:68–122, 1937.

[7] A. Baker. *Transcendental number theory*. Cambridge University Press, 1975.

[8] J. T. Baldwin and A. H. Lachlan. On strongly minimal sets. *Journal of Symbolic Logic*, 36:79–96, 1971.

[9] E. Barker. Back and forth relations for reduced Abelian p-groups. *Annals of Pure and Applied Logic*, 75:223–249, 1995.

[10] J. Barwise. *Admissible sets and structures*. Perspectives in Mathematical Logic. Springer-Verlag, 1975.

[11] A. S. Besicovitch. On the linear independence of fractional powers of integers. *Journal of the London Mathematical Society*, 15:3–6, 1940.

[12] W. Calvert. The isomorphism problem for classes of computable fields. *Archive for Mathematical Logic*, 43:327–336, 2004.

[13] W. Calvert. The isomorphism problem for computable Abelian p-groups of bounded length. accepted for publication in *Journal of Symbolic Logic*, 2005.

[14] W. Calvert, D. Cenzer, V. Harizanov, and A. Morozov. Δ^0_2-categoricity of Abelian p-groups. in preparation.

[15] W. Calvert, D. Cummins, J. F. Knight, and S. Miller. Comparing classes of finite structures. *Algebra and Logic*, 43:374–392, 2004.

[16] W. Calvert, V. Harizanov, J. F. Knight, and S. Miller. Index sets of computable structures. in preparation.

[17] W. Calvert, V. Harizanov, and A. Shlapentokh. Turing degrees of isomorphism types of algebraic objects. preprint, 2005.

[18] W. Calvert, J. F. Knight, and J. M. Young. Rank homogeneity and some consequenses. in preparation.

[19] D. A. Cox, J. B. Little, and D. B. O'Shea. *Using Algebraic Geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer Verlag, 1998.

[20] D. Cummins. Homogeneous embeddings. Senior Thesis, University of Notre Dame, 2003.

[21] R. G. Downey. On presentations of algebraic structures. In *Complexity, logic, and recursion theory*, volume 187 of *Lecture Notes in Pure and Applied Mathematics*, pages 157–205. Marcel Dekker, 1997.

[22] S. Feferman. Impredicativity of the existence of the largest divisible subgroup of an Abelian p-group. In *Model Theory and Algebra: A memorial tribute to Abraham Robinson*, number 498 in Lecture Notes in Mathematics, pages 117–130. Springer, 1975.

[23] S. Feferman and C. Spector. Incompleteness along paths in progressions of theories. *Journal of Symbolic Logic*, 27:383–390, 1962.

[24] H. Friedman, S. Simpson, and R. Smith. Countable algebra and set existence axioms. *Annals of Pure and Applied Logic*, 25:141–181, 1983.

[25] H. Friedman and L. Stanley. A Borel reducibility theory for classes of countable structures. *Journal of Symbolic Logic*, 54:894–914, 1989.

[26] A. Frolich and J. C. Sheperdson. Effective procedures in field theory. *Philosophical Transactions of the Royal Society of London, Series A*, 248:407–432, 1956.

[27] L. Fuchs. *Infinite Abelian groups*, volume 2. Academic Press, 1973.

[28] S. S. Goncharov. Autostability of models and Abelian groups. *Algebra and Logic*, 19:13–27, 1980.

[29] S. S. Goncharov, V. S. Harizanov, J. F. Knight, and R. A. Shore. Π^1_1 relations and paths through $\mathcal{O}$. *Journal of Symbolic Logic*, 69:585–611, 2004.

[30] S. S. Goncharov, V. S. Harizanov, M. C. Laskowski, S. Lempp, and C. F. D. McCoy. Trivial, strongly minimal theories are model complete after naming constants. *Proceedings of the American Mathematical Society*, 131:3901–3912, 2003.

[31] S. S. Goncharov and J. F. Knight. Computable structure and non-structure theorems. *Algebra and Logic*, 41:351–373, 2002.

[32] L. Harrington. Recursively presentable prime models. *The Journal of Symbolic Logic*, 39:305–309, 1974.

[33] J. Harrison. Recursive pseudo-well-orderings. *Transactions of the American Mathematical Society*, 131:526–543, 1968.

[34] D. Haskell, A. Pillay, and C. Steinhorn. *Model theory, algebra, and geometry.* Cambridge University Press, 2000.

[35] D. R. Hirschfeldt and W. M. White. Realizing levels of the hyperarithmetic hierarchy as degree spectra of relations on computable structures. *Notre Dame Journal of Formal Logic*, 43:51–64, 2002.

[36] G. Hjorth. *Classification and orbit equivalence relations.* American Mathematical Society, 1999.

[37] G. Hjorth. The isomorphism relation on countable torsion free Abelian groups. *Fundamenta Mathematicae*, 175:241–257, 2002.

[38] G. Hjorth and A. S. Kechris. Analytic equivalence relations and Ulm-type classifications. *Journal of Symbolic Logic*, 60:1273–1300, 1995.

[39] I. Kaplansky. *Infinite Abelian groups.* University of Michigan Press, 1969.

[40] N. G. Khisamiev. Strongly constructive models of a decidable theory. *Izvestija Akademii Nauk KazahskoĭSSR. Serija Fiziko-Matematičeskaja*, pages 83–84, 94, 1974.

[41] N. G. Khisamiev. Constructive Abelian p-groups. *Siberian Advances in Mathematics*, 2:68–113, 1992.

[42] J. F. Knight. Degrees coded in jumps of orderings. *The Journal of Symbolic Logic*, 51:1034–1042, 1986.

[43] J. F. Knight. Degrees of models. In *Handbook of Recursive Mathematics*, volume 1, pages 289–309. Elsevier, 1998.

[44] J. F. Knight. Effective transfer of invariants. preprint, 2005.

[45] J. F. Knight and J. M. Young. Computable structures of rank ω_1^{CK}. preprint, 2005.

[46] C. Lin. The effective content of Ulm's theorem. In *Aspects of effective algebra*, pages 147–160. Upside Down A Book Company, 1979.

[47] C. Lin. Recursively presented Abelian groups: effective p-group theory I. *Journal of Symbolic Logic*, 46:617 – 624, 1981.

[48] M. Makkai. An example concerning Scott heights. *The Journal of Symbolic Logic*, 46:301–318, 1981.

[49] A. I. Mal'cev. On recursive Abelian groups. *Dokl. Akad. Nauk SSSR*, 146:1009–1012, 1962.

[50] D. Marker. *Model theory: an introduction.* Springer-Verlag, 2002.

[51] W. E. Marsh. *On ω_1-categorical but not ω-categorical theories.* PhD thesis, Dartmouth College, 1966.

[52] G. Metakides and A. Nerode. Effective content of field theory. *Annals of Mathematical Logic*, 17:289–320, 1979.

[53] M. Morley. Categoricity in power. *Transactions of the American Mathematical Society*, 114:514–538, 1965.

[54] A. S. Morozov. Functional trees and automorphisms of models. *Algebra and Logic*, 32:28–38, 1993.

[55] M. E. Nadel. Scott sentences and admissible sets. *Annals of Mathematical Logic*, 7:267–294, 1974.

[56] M. E. Nadel. Scott heights of Abelian groups. *The Journal of Symbolic Logic*, 59:1351–1359, 1994.

[57] A. Nies. Undecidable fragments of elementary theories. *Algebra Universalis*, 35:8–33, 1996.

[58] A. Nurtazin. Strong and weak constructivizations and computable families. *Algebra and Logic*, 13:177–184, 1974.

[59] S. E. Oates. *Jump degrees of groups.* PhD thesis, University of Notre Dame, 1989.

[60] M. O. Rabin. Computable algebra, general theory and theory of computable fields. *Transactions of the American Mathematical Society*, 95:341–360, 1960.

[61] M. O. Rabin and D. Scott. The undecidability of some simple theories. preprint, 1962.

[62] F. Richman. The constructive theory of countable Abelian p-groups. *Pacific Journal of Mathematics*, 45:621 – 624, 1973.

[63] L. J. Richter. Degrees of structures. *The Journal of Symbolic Logic*, 46:723–731, 1981.

[64] L. Rogers. The structure of p-trees: algebraic systems related to Abelian groups. In *Abelian Group Theory: 2nd New Mexico State Conference*, pages 57–72. Springer-Verlag, 1976.

[65] D. B. Shapiro. Composites of algebraically closed fields. *Journal of Algebra*, 130:176–190, 1990.

[66] S. Shelah. *Classification theory and the number of non-isomorphic models.* North Holland, 1990.

[67] S. Simpson. *Subsystems of second order arithmetic*. Springer-Verlag, 1999.

[68] R. I. Soare. *Recursively enumerable sets and degrees*. Springer-Verlag, 1987.

[69] V. Stoltenberg-Hansen and J. V. Tucker. Computable rings and fields. In E. R. Griffor, editor, *Handbook of Computability Theory*, pages 363–447. Elsevier, 1999.

[70] S. Thomas. The classification problem for torsion-free Abelian groups of finite rank. *Journal of the American Mathematical Society*, 16, 2003.