Totally Magic Graphs

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Abstract

A *total labeling* of a graph with v vertices and e edges is defined as a one-to-one map taking the vertices and edges onto the integers $1, 2, \cdots, v+e$. Such a labeling is *vertex magic* if the sum of the label on a vertex and the labels on its incident edges is a constant independent of the choice of vertex, and *edge magic* if the sum of an edge label and the labels of the endpoints of the edge is constant. In this paper we examine graphs possessing a labeling that is simultaneously vertex magic and edge magic. Such graphs appear to be rare.

1 Introduction

All graphs in this paper are finite, simple and undirected. Unless otherwise specified, the graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we write e for $|E|$ and v for $|V|$. A general reference for graph theoretic notions is [6].

A *labeling* for a graph is a map that takes graph elements to numbers (usually positive or non-negative integers). In this paper the domain is the set of all vertices and edges; such labelings are called *total*. Other domains have also been studied. The most complete recent survey of graph labelings is [2].

In many cases, it is interesting to consider the sum of all labels associated with a graph label. This will be called the *weight* of the element. For example, the weight of vertex x under the total labeling λ is

$$wt(x) = \lambda(x) + \sum_{y \sim x} \lambda(xy),$$

while

$$wt(xy) = \lambda(x) + \lambda(xy) + \lambda(y).$$

If necessary, the labeling can be specified by a subscript, as in $wt_\lambda(x)$.

Various authors have introduced labelings that generalize the idea of a magic square, by requiring that certain weights be constant. Kotzig and Rosa [3], for example, defined a *magic labeling* to be a labeling on the vertices and edges in which the labels are the integers from 1 to $v + e$ and where the sum of labels on an edge and its two endpoints is constant. Related labelings have been studied by other authors and there are numerous variations in the terminology used. Readers are referred to [1] for a discussion of these matters and a standardization of the terminology.

In particular, we treat two types of total labeling. For detailed treatment of the two types taken separately, see [1] and [4].

An *edge magic total labeling* or EMTL on G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, v + e$, with the property that, given any edge xy,

$$\lambda(x) + \lambda(xy) + \lambda(y) = k$$

for some constant k. In other words, $wt(xy) = k$ for any edge xy.

A *vertex magic total labeling* or VMTL on G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, v + e$, with the property that, given any vertex x,

$$\lambda(x) + \sum_{y \sim x} \lambda(xy) = h$$

for some constant h, where the sum is over all vertices y adjacent to x. In this case the magic requirement is $wt(x) = h$ for all vertices x.

Suppose λ is a vertex magic total labeling of G. Then the sum of all vertex weights is

$$\begin{aligned} hv &= \sum_{x \in V} \lambda(x) + 2 \sum_{y \in E} \lambda(y) \\ &= (1 + 2 + \ldots + (v+e)) + \sum_{y \in E} \lambda(y) \\ &= \tfrac{1}{2}(v+e)(v+e+1) + \sum_{y \in E} \lambda(y). \end{aligned} \tag{1}$$

Similarly, if μ is an edge magic total labeling, then the sum of all edge weights is

$$\begin{aligned} ke &= \sum_{x \in V} d_x \mu(x) + \sum_{y \in E} \mu(y) \\ &= \tfrac{1}{2}(v+e)(v+e+1) + \sum_{x \in V} (d_x - 1)\mu(x), \end{aligned} \tag{2}$$

where d_x is the degree of x. As

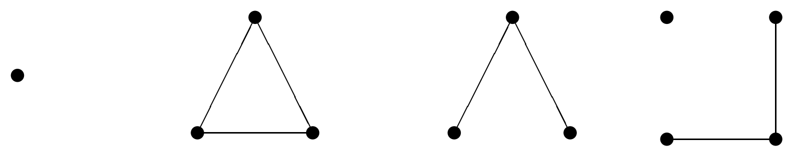
$$1 + 2 + \ldots + e \leq \sum_{y \in E} \lambda(y) \leq (v+1) + (v+2) + \ldots + (v+e),$$

(1) can be used to find upper and lower bounds for h. A similar approach, taking the degree sequence into account, yields upper and lower bounds for k.

In this paper we investigate the question: for a graph G does there exist a total labeling λ that is both edge magic and vertex magic? We shall call such a λ a *totally magic labeling* and call G a *totally magic* graph. The constants h and k will be called the *vertex constant* and *edge constant* respectively. We do not require that $h = k$.

2 Examples

One quickly constructs three small examples of connected totally magic graphs; see below.



K_1 $\quad C_3 \quad P_3 \quad K_1 \cup P_3$

An obvious trivial example is the single vertex graph K_1. There are four totally magic labelings of the triangle K_3; if the set of vertex-labels is denoted by S_v, then the labelings have

$$\begin{aligned} h &= \ \ 9,\ k = 12,\ S_v = \{4, 5, 6\} \\ h &= 10,\ k = 11,\ S_v = \{2, 4, 6\} \\ h &= 11,\ k = 10,\ S_v = \{1, 3, 5\} \\ h &= 12,\ k = \ \ 9,\ S_v = \{1, 2, 3\}. \end{aligned}$$

(There is an obvious duality here.) The three-vertex path P_3 has two labelings. Writing the labels in sequence vertex-edge-vertex-edge-vertex, they are

$$\begin{aligned} h &= 6,\ k = 9,\ \text{ labels } 4, 2, 3, 1, 5, \\ h &= 7,\ k = 8,\ \text{ labels } 3, 4, 1, 2, 5. \end{aligned}$$

Among disconnected graphs, we know that there is exactly one totally magic labeling of $K_1 \cup P_3$, constructed from the first of the P_3-labelings listed above by mapping the isolated vertex to 6.

3 Isolates and stars

If λ is a totally magic labeling of G with vertex constant h then any isolated vertex x has $\lambda(x) = h$, so there cannot be two such vertices. Similarly, an isolated edge would have equal labels on its endpoints. We have:

Lemma 1 *No totally magic graph has two isolated vertices or an isolated edge.*

Moreover, if $K_1 \cup G$ is totally magic, the isolated vertex must necessarily receive the largest possible label, so the remaining labels form a totally magic labeling of G. We have:

Lemma 2 *If a graph with an isolated vertex is totally magic, then the graph G resulting from the deletion of the isolate has a totally magic labeling with vertex sum $|V(G)| + |E(G)| + 1$.*

The labeling of $K_1 \cup P_3$ given above is as described in this lemma.

A vertex of degree 1 is often called a *leaf.*

Theorem 3 *Suppose the totally magic graph G has a leaf x. Then the component of G containing x is a star.*

Proof. Suppose λ is a totally magic labeling on G, with vertex and edge constants h and k respectively, and suppose x is a leaf with neighbor y. By the vertex magic property, $\lambda(x) + \lambda(xy) = h$, and by the edge magic property, $\lambda(x) + \lambda(xy) + \lambda(y) = k$. So $\lambda(y) = k - h$.

By Lemma 1, y has a neighbor, z say. Then $k = \lambda(y) + \lambda(yz) + \lambda(z) = \lambda(yz) + \lambda(z) + k - h$, so $\lambda(yz) = h - \lambda(z)$. So $wt(z) \geq \lambda(z) + \lambda(yz) = h$, with equality only if z has degree 1. So every vertex adjacent to y has degree 1, and the component of G containing y is a star with center y. □

Corollary 3.1 *The only connected totally magic graph containing a vertex of degree 1 is P_3.*

Every non-trivial tree has at least two vertices of degree 1, so the only possible magic trees are K_1 and the stars. But it was shown in Theorem 5 of [4] that $K_{m,n}$ is never vertex magic when $|m - n| > 1$. So no star larger than $K_{1,2}$ is vertex magic, so

Corollary 3.2 *The only totally magic trees are K_1 and P_3.*

A totally magic graph cannot have two stars as components, because their centers would each receive label $k - h$. It follows that the components of a totally magic graph can include at most one K_1 and at most one star, and all other components have minimum degree at least 2, and consequently have as many edges as vertices.

Corollary 3.3 *The only totally magic proper forest is $K_1 \cup P_3$.*

Theorem 4 *The only totally magic graphs with a component K_1 are $K_1 \cup P_3$ and K_1 itself.*

Proof. Suppose $K_1 \cup G$ is a totally magic graph; K_1 has vertex x, and G has v vertices and e edges, as usual. G may have a star as a component, but any other component has minimum degree at least 2, so $e \geq v - 1$. From Lemma 2,

$$h = \lambda(x) = v + e + 1,$$

and G is totally magic with vertex constant h. Now from (1),

$$v(v + e + 1) = \tfrac{1}{2}(v + e)(v + e + 1) + \sum_{y \in E} \lambda(y).$$

So

$$\sum_{y \in E} \lambda(y) = \tfrac{1}{2}(v - e)(v + e + 1).$$

But $\sum \lambda(y) \geq \frac{1}{2}e(e + 1)$, so

$$e(e + 1) \leq (v - e)(v + e + 1),$$

and

$$2e(e + 1) \leq v(v + 1).$$

Clearly $e < v$. So $e = v - 1$, whence

$$2(v^2 - v) \leq v^2 + v,$$

and $v \leq 3$. The only possibility is $G = P_3$. $\square$

4 Forbidden configurations

Theorem 5 *If a totally magic graph G contains two adjacent vertices of degree 2, then the component containing them is a cycle of length 3.*

Proof. Suppose G contains a pair of adjacent vertices b and c, each having degree 2, and suppose λ is a totally magic labeling of G with vertex and edge constants h and k.

First, assume G contains a path $\{a, b, c, d\}$, where a and d are distinct vertices. From $h = wt(b) = wt(c)$ it follows that

$$\lambda(ab) + \lambda(b) + \lambda(bc) = \lambda(bc) + \lambda(c) + \lambda(cd),$$

from which

$$\lambda(cd) = \lambda(ab) + \lambda(b) - \lambda(c) \tag{3}$$

while the edge magic property yields

$$k = \lambda(a) + \lambda(ab) + \lambda(b) = \lambda(b) + \lambda(bc) + \lambda(c) = \lambda(c) + \lambda(cd) + \lambda(d); \quad (4)$$

the second equality in (4) implies

$$\lambda(cd) = \lambda(b) + \lambda(bc) - \lambda(d) \quad (5)$$

while the first gives

$$\lambda(a) + \lambda(ab) = \lambda(bc) + \lambda(c). \quad (6)$$

But (3) and (5) give

$$\lambda(ab) - \lambda(c) = \lambda(bc) - \lambda(d)$$

so

$$\lambda(d) = \lambda(bc) + \lambda(c) - \lambda(ab) \quad (7)$$

and (6) and (7) together imply $\lambda(a) = \lambda(d)$, a contradiction.

Now assume b and c have a common neighbor, a say, and suppose a has some other neighbor, z. ¿From (4) we see

$$\lambda(ab) = k - \lambda(a) - \lambda(b),$$

and similarly for bc and ca, so

$$\begin{aligned} wt(b) &= \lambda(ab) + \lambda(b) + \lambda(bc) \\ &= 2k - \lambda(a) - \lambda(b) - \lambda(c), \end{aligned}$$

whereas

$$\begin{aligned} wt(a) &= \lambda(ca) + \lambda(a) + \lambda(ab) + \lambda(az) + \ldots \\ &\geq 2k - \lambda(a) - \lambda(b) - \lambda(c) + \lambda(az) \\ &> wt(a), \end{aligned}$$

as $\lambda(az) > 0$. □

Corollary 5.1 *No totally magic graph contains as a component a path other than P_3 or a cycle other than K_3.*

(Lemma 1 must be invoked to rule out P_2.) In particular,

Corollary 5.2 *The only totally magic cycle is K_3.*

Theorem 6 *Suppose G contains two vertices, x_1 and x_2, that are each adjacent to precisely the same set $\{y_1, y_2, \ldots, y_d\}$ of other vertices. (It is not specified whether x_1 and x_2 are adjacent.) If $d > 1$ then G is not totally magic.*

Proof. Suppose λ is a totally magic labeling of G with vertex and edge sums h and k. For convenience, define $\lambda(x_1x_2) = 0$ if x_1 is not adjacent to x_2. Then

$$h = \lambda(x_i) + \sum_{j=1}^{d} \lambda(x_iy_j) + \lambda(x_1x_2),\ i = 1, 2,$$

and

$$k = \lambda(x_i) + \lambda(x_iy_j) + \lambda(y_j),\ i = 1, 2,\ 1 \leq j \leq d.$$

So

$$\begin{aligned} dk &= d\lambda(x_i) + \sum_{j=1}^{d} \lambda(x_iy_j) + \sum_{j=1}^{d} \lambda(y_j),\ i = 1, 2, \\ &= (d-1)\lambda(x_i) + (h - \lambda(x_1x_2)) + \sum_{j=1}^{d} \lambda(y_j),\ i = 1, 2, \end{aligned}$$

so

$$(d-1)\lambda(x_1) = (d-1)\lambda(x_2),$$

a contradiction unless $d = 1$. □

Corollary 6.1 *The only totally magic complete graphs are K_1 and K_3. The only totally magic complete bipartite graph is $K_{1,2}$.*

Theorem 7 *Suppose G contains two vertices, x and y, with a common neighbor. If x and y are nonadjacent and each has degree 2, or are adjacent and each has degree 3, then G is not totally magic.*

Proof. Denote the common neighbor by z, the other neighbor of x by x_1, and the other neighbor of y by y_1. (Possibly $x_1 = y_1$.) Suppose λ is a totally magic labeling of G with vertex and edge constants h and k; if x is not adjacent to y in G then define $\lambda(xy) = 0$.

The weight of x is

$$\begin{aligned} wt(x) &= \lambda(x) + \lambda(xx_1) + \lambda(xz) + \lambda(xy) \\ &= \lambda(x) + (k - \lambda(x) - \lambda(x_1)) + (k - \lambda(x) - \lambda(z)) + \lambda(xy) \\ &= 2k - (\lambda(x) + \lambda(x_1) + \lambda(z)) + \lambda(xy) \end{aligned}$$

and similarly

$$wt(y) = 2k - (\lambda(y) + \lambda(y_1) + \lambda(z)) + \lambda(xy).$$

From the vertex magic property, $wt(x) = wt(y)$, so $\lambda(x)+\lambda(x_1) = \lambda(y)+\lambda(y_1)$. So $\lambda(xx_1) = k-\lambda(x)-\lambda(x_1) = k-\lambda(y)-\lambda(y_1) = \lambda(yy_1)$. But this contradicts the edge magic property. □

(The case where x and y are adjacent could be rephrased, a totally magic graph G cannot contain a triangle with two vertices of degree 3.)

Theorem 8 *Suppose the totally magic graph G contains a triangle. Then the sum of the labels of all edges outside the triangle and incident with any one vertex of the triangle is the same, whichever vertex is chosen.*

Proof. Suppose the triangle is xyz. Write X for the sum of the labels of all edges other than xy and xz that are adjacent to x; define Y and Z similarly. Then

$$\begin{aligned} h &= wt(x) = \lambda(x) + \lambda(xy) + \lambda(xz) + X = 2k - \lambda(x) - \lambda(y) - \lambda(z) + X \\ &= wt(y) = \lambda(y) + \lambda(xy) + \lambda(yz) + Y = 2k - \lambda(x) - \lambda(y) - \lambda(z) + Y \\ &= wt(z) = \lambda(z) + \lambda(xz) + \lambda(yz) + Z = 2k - \lambda(x) - \lambda(y) - \lambda(z) + Z \end{aligned}$$

so $X = Y = Z$. □

Corollary 8.1 *If the totally magic graph G contains a triangle with one vertex of degree 2, then the triangle is a component of G.*

Observe that Theorems 5, 6, 7 and 8 are essentially *forbidden configuration* theorems. If a graph G is in violation of one of them, then not only is G not totally magic, but G cannot be a component or union of components in any totally magic graph. We term a graph a *survivor* if it is not eliminated by the application of these theorems.

5 Unions of triangles

In this section we construct two infinite families of totally magic graphs, both based on triangles. We use the following Lemma, which has been proved several times in the literature.

Lemma 9 *The set of integers $\{1, 2, \cdots, 3n\}$ can be partitioned into n triples, where the sum of elements in any triple is $\frac{3}{2}(3n+1)$, if and only if n is odd.*

Proof. Observe that $1 + 2 + \ldots + 3n = 3n(3n + 1)/2$, so the only possible common sum is $\frac{3}{2}(3n + 1)$.

First suppose n is odd. For $i = 1, 2, \ldots, n$, take x_i to be the integer such that $x_i = 3n - 2i + 2$ and $2n + 1 \leq x_i \leq 3n$, and take y_i to be the integer such that $y_i = (3n+2i-1)/2$ and $n+1 \leq y_i \leq 2n$. Then one solution is $\{S_1, S_2, \ldots, S_n\}$, where $S_i = \{i, x_i, y_i\}$.

If n is even, then $\frac{3}{2}(3n+1)$ is not an integer. $\square$

Theorem 10 *There is a totally magic labeling of nK_3, the disjoint union of n triangles, whenever n is odd.*

Proof. Choose n 3-sets as defined in Lemma 9. For each triangle, label the edges with the members of the triple. If edge xy receives label $\lambda(xy)$, the vertex opposite xy receives $3n+\lambda(xy)$. As the three edge-labels on a triangle sum to $\frac{3}{2}(3n+1)$, the edges all have weight $6n+\frac{3}{2}(3n+1)$ and the vertices all have weight $3n+\frac{3}{2}(3n+1)$. $\square$

In the above construction, label 1 appears on an edge. If this edge is deleted, and 1 is subtracted from the label of each vertex and each remaining edge, the resulting labeling is totally magic, with vertex constant $3n+\frac{3}{2}(3n-1)$ and edge constant $6n+\frac{3}{2}(3n-1)$. This produces another infinite family:

Corollary 10.1 *The graph $P_3 \cup nK_3$ is totally magic when n is even.*

On the other hand, an even union of triangles is never totally magic.

Theorem 11 *There is no totally magic labeling of nK_3, the disjoint union of n triangles, whenever n is even.*

Proof. Suppose λ is a totally magic labeling of nK_3, where $n=2m$ for some positive integer m. Write h and k for the vertex and edge constants of λ. If xyz is a triangle, write $\mu(x)$ for $\lambda(yz)$.

Summing the weights of edges and vertices, we have $\sum_x(\lambda(x)+2\mu(x)) = 6mh$ and $\sum_x(\mu(x)+2\lambda(x)) = 6mk$, so $\sum_x(\lambda(x)+\mu(x)) = 2m(h+k)$. Since the labels are $1, 2, \ldots, 12m$,

$$36m+3 = h+k. \tag{8}$$

Consider a triangle xyz. Clearly $\lambda(xy) = k-\lambda(x)-\lambda(y)$. So $\lambda(x)+\lambda(xy)+\lambda(xz) = h$ implies $\lambda(x)+k-\lambda(x)-\lambda(y)+k-\lambda(x)-\lambda(z) = h$. So the sum of the three vertex labels is the same in any triangle:

$$\lambda(x)+\lambda(y)+\lambda(z) = 2k-h.$$

Similarly the sum of the three edge labels in a triangle is constant:

$$\lambda(xy)+\lambda(yz)+\lambda(zx) = 2h-k.$$

So $\lambda(xy) = k-\lambda(x)-\lambda(y) = k+\lambda(z)-\lambda(x)-\lambda(y)-\lambda(z) = h-k+\lambda(z)$, or

$$\lambda(xy) = h-k+\lambda(x). \tag{9}$$

Let us assume that $h \geq k$ (otherwise, we can exchange the labels of the vertices and the edges and obtain another totally magic labeling with $h \geq k$). Then from (9) labels $1, 2, \ldots, h-k$ must all be vertex labels, and $h-k+1, h-k+2, \ldots, 2(h-k)$ are edge labels. So $2(h-k)+1, 2(h-k)+2, \ldots, 3(h-k)$ are vertex labels, and so on: the vertex labels are precisely the labels a satisfying $a \equiv 1, 2, \ldots, (h-k)(\bmod 2(h-k))$. Since there are $6m$ vertices and $6m$ edges, $h-k$ divides $6m$. Clearly $h-k \equiv h+k = 36m+3 \equiv 1(\bmod 2)$, so , $h-k$ is odd, $h-k$ divides $3m$, and therefore $1 \leq h-k \leq 3m$.

First, suppose $h-k=1$. We have to partition the edge labels $2, 4, \ldots, 12m$ into $2m$ sets of size 3 such that the sum of the labels in each set is constant. If such a partition exists, halving it provides a partition of $1, 2, \ldots, 6m$ into $2m$ sets of size 3 such that the sum of the labels in each set is a constant. But such a partition is impossible by Lemma 9.

So we assume $h-k \geq 3$. Write $h-k = 2c+1$. From (8) we obtain $h = 18m+c+2$ and $k = 18m-c+1$. So $2k-h = 18m-3c$, and $2k-h \equiv 18m-3c \equiv -3c \equiv c+2(\bmod 4c+2)$, since $2c+1$ divides $3m$.

Each vertex label must be congruent to one of $1, 2, \ldots, h-k(\bmod 2(h-k))$, that is $1, 2, \ldots, 2c+1(\bmod 4c+2)$. The sum of the three vertex labels in any triangle will be congruent to $2k-h$. Now, $2k-h \equiv c+2(\bmod 4c+2)$. Since $0 < c < h-k, c+1$ must be a vertex label. But this is impossible, since there do not exist two elements $d, e \in \{1, 2, \ldots, 2c+1\}(\bmod 4c+2)$ such that $c+1+d+e \equiv c+2(\bmod 4c+2)$.

Hence $2mK_3$ is not totally magic for any positive integer m. □

If there were a totally magic labeling of $P_3 \cup nK_3$ with vertex and edge constants h and k, it is easy to show that the labels on the endpoints of the P_3 must add to k. Suppose those two endpoints are joined, and the new edge is labeled 0. Then the labeling of $(n+1)K_3$, constructed by adding 1 to every label, will be totally magic. From Theorem 11, this cannot occur when n is odd. So

Corollary 11.1 *The graph $P_3 \cup nK_3$ is not totally magic when n is odd.*

6 Small graphs

We have carried out a complete search for small totally magic graphs. The result is that no examples with ten or fewer vertices exist, other than the four graphs given in Section 2 and the two nine-vertex graphs constructed in Theorem 10 and Corollary 10.1.

The complete search was carried out in two stages. First, using *nauty* [5], lists were prepared of all connected graphs (up to ten vertices) not ruled out by Theorems 5, 6, 7 and 8. Second, the survivors were tested exhaustively. There were three survivors with six or fewer vertices (K_1, K_3 and P_3), 42 with seven, 1,070 with eight, 61,575 with nine and 4,579,637 with 10.

Exhaustive testing is very time-consuming. However, a shortcut is available. If a totally magic labeling exists, it must retain the magic properties after reduction modulo 2. So we tested all mod 2 possibilities. Label 1 or 0 is assigned to each vertex and to the constant k. Then every edge-label can be calculated. Next, one can check whether all the vertex weights are congruent (mod 2). Moreover, the total number of vertices and edges labeled 1 must either equal the number labeled 0 or exceed that number by 1. This process is quite fast (for example, only 2^9 cases need to be examined in the eight-vertex case), and eliminated over 25% of graphs. Then one can sieve the remaining graphs modulo 3, then modulo 4, and so on. For example, sieving the 1070 eight-vertex graphs mod 2 eliminated 307 graphs, sieving mod 3 eliminated 351 more, and so on: one graph survived after sieving modulo 7, and it was eliminated mod 8.

There were very few disconnected graphs to consider. Except for the graph $K_1 \cup P_3$, the only possibilities are made up of at most one star, copies of K_3, and survivors with more than three vertices. The only cases not already discussed are $K_{1,n} \cup K_3$ for $n = 3, 4, 5, 6, K_{1,4} \cup 2K_3$, and the 42 unions of a triangle and a 7-vertex survivor. None of these is totally magic, so there are no further totally magic graphs with ten or fewer vertices.

There are exactly eight totally magic labelings of $3K_3$. They arise in dual pairs (the dual is derived by exchanging the label of each vertex with that of its opposite edge). Of the four pairs, two come from the construction of Theorem 10 and two do not. In each dual pair, one member has 1 as an edge label, so there are four totally magic labelings of $P_3 \cup 2K_3$.

We carried out a further investigation using a variant of simulated annealing. This procedure quickly found the graphs we have described, but has so far found no other examples. This might suggest that we have found all totally magic graphs. However, for larger numbers of vertices (more than 20, say), it appears that the search gets "nearer" to satisfaction (no, we do not wish to clarify this vague description!), so perhaps there are large totally magic graphs yet to be discovered.

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