Double Arrays, Triple Arrays and Balanced Grids

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Abstract

Triple arrays are a class of designs introduced by Agrawal in 1966 for two-way elimination of heterogeneity in experiments. In this paper we investigate their existence and their connection to other classes of designs, including balanced incomplete block designs and balanced grids.

1 Combinatorial designs in general

We shall follow the standard notations for combinatorial designs, as for example are laid down by Preece in the survey [11]. For convenience we restate the definitions that will be most important in this paper.

A *balanced incomplete block design* with parameters v, b, r, k, λ, denoted a (v, b, r, k, λ)-$BIBD$, is a way of selecting b subsets of size k, or *blocks*, from some v-set $V = \{t_1, t_2, \ldots, t_v\}$ of *treatments*, in such a way that every treatment appears in exactly r blocks (the design is *equireplicate*) and any two treatments appear together in exactly λ blocks (the design is *balanced*). The set V is called the *support* of the design. Since the blocks are defined as sets, these designs are necessarily *binary* (that is, there are no repeated members in blocks). The parameters of balanced incomplete block designs satisfy the two conditions

$$bk = vr, \tag{1}$$
$$\lambda(v-1) = r(k-1), \tag{2}$$

(see, for example, [19]). Without loss of generality, we usually write $V = \{1, 2, \ldots, v\}$. The *complement* of a (v, b, r, k, λ)-$BIBD$ is the $(v, b, b-r, v-k, b-2r+\lambda)$-$BIBD$ formed by replacing each block B by its complement in V, $V \backslash B$.

A balanced incomplete block design is called *symmetric* if $v = b$ (or, equivalently, $r = k$); in such a design, any two blocks intersect in λ elements. A symmetric balanced incomplete block design with parameters (v, k, λ) is often called a (v, k, λ)-$SBIBD$. Among symmetric balanced incomplete block designs are the *finite projective planes*, the symmetric designs with $\lambda = 1$, which exist if (and, it has been conjectured, only if) $k-1$ is a prime power. Given a (v, k, λ)-$SBIBD$, its *residual design* modulo a given block B is constructed by deleting B from the list of blocks and deleting every member of B from the remaining blocks. The *derived design* modulo B is formed by deleting B and, in the remaining blocks, retaining *only* the members of B. The residual and derived designs are a $(v-k, v-1, k, k-\lambda, \lambda)$-$BIBD$ and a $(k, v-1, k-1, \lambda, \lambda-1)$-$BIBD$ respectively. Stanton [17] pointed out that the complement of the derived design of a symmetric balanced incomplete block design $\mathcal{B}$ equals the residual of the complement of $\mathcal{B}$.

The usual definition of a balanced incomplete block design requires that $k < v$ ("the blocks are incomplete"), but we shall find it convenient to allow the trivial case where $k = v$.

A binary *row-column design* is a rectangular array whose entries are members of some set of *treatments*, with no repetitions in any row or column. If such a design has r rows, c columns, and v treatments which form a v-set V, then it is an $r \times c$ *binary row-column design based on* V. We normally use the same notational conventions for V as above.

A binary row-column design is called *equireplicate* if every member of V appears the same number of times in the array. This common number is then called the *replication number* of the design.

Among binary row-column designs, perhaps the best-known is the *Latin square*, the designs with $r = c = v$, which are easily seen to exist for all values of v. Another important class are *Youden squares.* A Youden square is a $k \times v$ array based on a (v, k, λ)-$SBIBD$. Each column contains the elements of one block, ordered so that each element appears exactly once in each row. It was shown in [16] that such an ordering is always possible; that is, every symmetric balanced incomplete block design gives rise to a Youden square. (In fact, it is common for many non-isomorphic Youden squares to arise from the same $SBIBD$.)

When discussing binary row-column designs, we shall often need to refer to the set of all elements of a row, ignoring the arrangement of the elements into columns. It will be convenient to extend the usage in design theory and refer to this set as the *support* of the row; and similarly for columns.

2 Double arrays

We wish to investigate a class of binary row-column designs that were defined by Agrawal [], although a small example was discussed earlier by Potthoff [9] and another was published by Preece [10] independently of Agrawal's paper. We shall introduce these designs below under the name of *triple arrays*. It will be convenient to begin by introducing a more general class, *double arrays*.

Suppose $\mathcal{A}$ is an equireplicate $r \times c$ binary row-column design based on V, with replication number k, having the following properties:

(P1) any two distinct rows have the same number, λ_{rr}, of common elements;

(P2) any two distinct columns have the same number, λ_{cc}, of common elements.

Then $\mathcal{A}$ is a *double array* with parameters $v, k, \lambda_{rr}, \lambda_{cc}$, or

$$DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c).$$

Associated with any double array are two balanced incomplete block designs. To construct them, suppose the rows of a $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$ are labeled $R_1, R_2, \ldots, R_r$ and the columns are labeled $C_1, C_2, \ldots, C_c$. Then the *row design* or $BIBD_R$ has v blocks $B_1, B_2, \ldots, B_v$, corresponding to the v elements of V: if element x appears in rows $R_a, R_b, \ldots, R_z$ then $B_x = \{a, b, \ldots, z\}$. Similarly the *column design* or $BIBD_C$ is defined using the incidence of elements in columns.

Lemma 2.1 *Suppose $\mathcal{A}$ is a $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$. Then*

(i) *the row design of $\mathcal{A}$ is a balanced incomplete block design with parameters*

$$(r, v, c, k, \lambda_{rr}).$$

(ii) *the column design of $\mathcal{A}$ is a balanced incomplete block design with parameters*

$$(c, v, r, k, \lambda_{cc}).$$

Theorem 2.2 *Any* $DA(v,k,\lambda_{rr},\lambda_{cc}:r\times c)$ *satisfies*

$$\begin{aligned} vk &= rc, & (3)\\ \lambda_{rr}(r-1) &= c(k-1), & (4)\\ \lambda_{cc}(c-1) &= r(k-1), & (5)\\ \lambda_{rr}r(r-1) &= \lambda_{cc}c(c-1). & (6) \end{aligned}$$

Proof. Equation (3) follows from applying (1) to either of the designs associated with $\mathcal{A}$. Equations (4) and (5) are just (2), for the $BIBD_R$ and $BIBD_C$ respectively. Equation (6) is obtained by combining (4) and (5). □

Example. Here is a $DA(10,3,3,2:5\times 6)$:

	C_1	C_2	C_3	C_4	C_5	C_6
R_1	1	2	3	4	5	6
R_2	4	7	1	3	8	9
R_3	2	5	10	8	9	3
R_4	10	8	7	6	1	2
R_5	9	4	5	10	6	7

. (7)

The $BIBD_R$ has parameters $(5,10,6,3,3)$, with blocks

$$124, 134, 123, 125, 135, 145, 245, 234, 235, 345,$$

while the $BIBD_C$ has parameters $(6,10,5,3,2)$ and blocks

$$135, 126, 346, 124, 235, 456, 236, 245, 156, 134.$$

Clearly any double array must satisfy $1\le k\le r$ and $1\le k\le c$. The extreme cases of double arrays need to be discussed. Both extreme cases will be called *trivial*.

If $k=1$, so that no entry of the array is repeated, then $\lambda_{rr}=\lambda_{cc}=0$, and the array is a $DA(rc,1,0,0:r\times c)$. This design exists for every r and c.

If $k=r$, (3) yields $c=v$, so every row is a permutation of the set V. So the array is a Latin rectangle. The $BIBR_R$ is trivial, and the $BIBD_C$ is a (v,r,λ_{cc})-$SBIBD$. As we said previously, such a design is called a *Youden square* and it is well-known that such a design exists if and only if there is a (v,r,λ_{cc})-$SBIBD$. (Many Youden squares may correspond to the same $SBIBD$.) The case $k=c$ yields the transpose of a Youden square.

3 Triple arrays

Suppose $\mathcal{A}$ is a double array with parameters $(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$. Suppose further that $\mathcal{A}$ satisfies the following condition:

(P3) any row and any column have the same number, λ_{rc}, of common elements.

Then $\mathcal{A}$ is called a *triple array* with parameters $v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc}$, or

$$TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c).$$

Not every double array is a triple array. The smallest example is

$$\begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 3 & 1 & 5 & 6 \\ 6 & 4 & 2 & 5 \\ \hline \end{array},$$

which is a $DA(6, 2, 2, 1 : 3\times4)$ but is not a triple array. In fact, an exhaustive search shows that there is no 3×4 triple array. In Section 7 we shall describe a $DA(24, 6, 10, 3 : 9\times16)$ which is not a triple array, and cannot be transformed into a triple array by any sequence of permutations within its columns.

Theorem 3.1 *In any $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$,*

$$\lambda_{rc} = k. \tag{8}$$

Proof. Write I_{ij} for the intersection of row i with column j. If x is any particular entry in $\mathcal{A}$, the binary property means that x belongs to k of the rows and k of the columns, so it is in k^2 of the sets I_{ij}. As there are v different entries, the total number of entries in the I_{ij} is k^2v. But each I_{ij} is a λ_{rc}-set and there are rc of them. So $k^2v = rc\lambda_{rc}$. But $vk = rc$ from (3), and the result follows. □

It is easy to see that any trivial double array is also a triple array (in both cases $\lambda_{rc} = k$), and these will be called *trivial triple arrays*. More interestingly, the example (7) is a triple array, a $TA(10, 3, 3, 2, 3 : 5 \times 6)$.

Theorem 3.2 *Any triple array with $k \neq r$ and $k \neq c$ satisfies*

$$v \geq r + c - 1. \tag{9}$$

Proof. Suppose $\mathcal{A}$ is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ where $k \neq r$. Let R and C denote the incidence matrices of the corresponding $BIBD_R$ and $BIBD_C$ respectively. The $r \times v$ matrix R satisfies $RR^T = (c - \lambda_{rr})I + \lambda_{rr}J_{rr}$ and the $c \times v$ matrix C satisfies $CC^T = (r - \lambda_{cc})I + \lambda_{cc}J_{cc}$, where J_{pq} denotes the $p \times q$ matrix with every entry 1. From the definition of λ_{rc} it follows that $RC^T = \lambda_{rc}J_{rc} = kJ_{rc}$ and $CR^T = \lambda_{rc}J_{cr} = kJ_{cr}$.

The $(r + c) \times v$ matrix A is defined by

$$A = \begin{bmatrix} R \\ C \end{bmatrix}.$$

Then AA^T is $(r + c) \times (r + c)$, and satisfies

$$\begin{aligned} AA^T &= \begin{bmatrix} R \\ C \end{bmatrix} \begin{bmatrix} R^T \; C^T \end{bmatrix} \\ &= \begin{bmatrix} RR^T \; RC^T \\ CR^T \; CC^T \end{bmatrix} \\ &= \begin{bmatrix} (c - \lambda_{rr})I + \lambda_{rr}J_{rr} & kJ_{rc} \\ kJ_{cr} & (r - \lambda_{cc})I + \lambda_{cc}J_{cc} \end{bmatrix}. \end{aligned} \tag{10}$$

We show that AA^T has rank $r + c - 1$. Then

$$r + c - 1 = \operatorname{rank}(AA^T) \leq \operatorname{rank}(A) \leq v.$$

To find the rank we use row and column reduction.

$$AA^T = \left[\begin{array}{cccc|cccc} c & \lambda_{rr} & \dots & \lambda_{rr} & k & k & \dots & k \\ \lambda_{rr} & c & \dots & \lambda_{rr} & k & k & \dots & k \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{rr} & \lambda_{rr} & \dots & c & k & k & \dots & k \\ \hline k & k & \dots & k & r & \lambda_{cc} & \dots & \lambda_{cc} \\ k & k & \dots & k & \lambda_{cc} & r & \dots & \lambda_{cc} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ k & k & \dots & k & \lambda_{cc} & \lambda_{cc} & \dots & r \end{array}\right]$$

Subtracting column 1 from columns 2, 3, ..., r, and column $r+1$ from the later columns, we obtain

$$\left[\begin{array}{cccccc|ccccc}
c & \lambda_{rr}-c & \lambda_{rr}-c & \dots & \lambda_{rr}-c & k & 0 & 0 & \dots & 0 \\
\lambda_{rr} & c-\lambda_{rr} & 0 & \dots & 0 & k & 0 & 0 & \dots & 0 \\
\lambda_{rr} & 0 & c-\lambda_{rr} & \dots & 0 & k & 0 & 0 & \dots & 0 \\
\vdots & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\lambda_{rr} & 0 & 0 & \dots & 0 & k & 0 & 0 & \dots & 0 \\
\lambda_{rr} & 0 & 0 & \dots & c-\lambda_{rr} & k & 0 & 0 & \dots & 0 \\
\hline
k & 0 & 0 & \dots & 0 & r & \lambda_{cc}-r & \lambda_{cc}-r & \dots & \lambda_{cc}-r \\
k & 0 & 0 & \dots & 0 & \lambda_{cc} & r-\lambda_{cc} & 0 & \dots & 0 \\
k & 0 & 0 & \dots & 0 & \lambda_{cc} & 0 & r-\lambda_{cc} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \vdots \\
k & 0 & 0 & \dots & 0 & \lambda_{cc} & 0 & 0 & \dots & 0 \\
k & 0 & 0 & \dots & 0 & \lambda_{cc} & 0 & 0 & \dots & r-\lambda_{cc}
\end{array}\right]$$

Now add rows 2, 3, ..., r to row 1, and rows $r+1, r+2, r+3, \dots, r+c$ to row r. The $(1,1)$ entry becomes $c+(r-1)\lambda_{rr}$; by (4) this equals $c+c(k-1)$, which equals ck. Similarly, the $(r+1, r+1)$ entry equals rk.

$$\left[\begin{array}{cccccc|ccccc}
ck & 0 & 0 & \dots & 0 & rk & 0 & 0 & \dots & 0 \\
\lambda_{rr} & c-\lambda_{rr} & 0 & \dots & 0 & k & 0 & 0 & \dots & 0 \\
\lambda_{rr} & 0 & c-\lambda_{rr} & \dots & 0 & k & 0 & 0 & \dots & 0 \\
\vdots & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\lambda_{rr} & 0 & 0 & \dots & 0 & k & 0 & 0 & \dots & 0 \\
\lambda_{rr} & 0 & 0 & \dots & c-\lambda_{rr} & k & 0 & 0 & \dots & 0 \\
\hline
ck & 0 & 0 & \dots & 0 & rk & 0 & 0 & \dots & 0 \\
k & 0 & 0 & \dots & 0 & \lambda_{cc} & r-\lambda_{cc} & 0 & \dots & 0 \\
k & 0 & 0 & \dots & 0 & \lambda_{cc} & 0 & r-\lambda_{cc} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \vdots \\
k & 0 & 0 & \dots & 0 & \lambda_{cc} & 0 & 0 & \dots & 0 \\
k & 0 & 0 & \dots & 0 & \lambda_{cc} & 0 & 0 & \dots & r-\lambda_{cc}
\end{array}\right]$$

If row $r+1$ and column $r+1$ are deleted from this matrix, the resulting $(r+c-1)\times(r+c-1)$ matrix has determinant $ck(c-\lambda_{rr})^{r-1}(r-\lambda_{cc})^{c-1}$. If $c-\lambda_{rr}=0$ then (4) gives $k=r$, which is not allowed, and similarly

$r - \lambda_{cc} = 0$ gives $k = c$. So the determinant is non-zero, and the matrix AA^T has rank at least $r+c-1$. But rows 1 and $r+1$ are identical, so AA^T cannot have rank $r+c$. So it has rank $r+c-1$, as required. □

Question: can we ever get $v < r + c - 1$ in a **double** *array?*

4 Balanced grids

We now introduce another type of binary row-column design. Its relevance will be seen in Section 6.

Suppose $\mathcal{G}$ is any binary row-column design. We shall define μ_{xy} to be the number of times that elements x and y occur in the same row or column of $\mathcal{G}$. In other words, if there are r_1 rows that contain both x and y, and c_1 columns that contain them both, then $\mu_{xy} = r_1 + c_1$. A binary row-column design will be called a *balanced grid* if there is a constant μ such that $\mu_{xy} = \mu$ for every x and y.

Theorem 4.1 *An $r \times c$ balanced grid based on v symbols satisfies*

$$\mu = \frac{rc(r+c-2)}{v(v-1)}; \tag{11}$$

moreover, it will be equireplicate, with replication number

$$k = \frac{rc}{v}. \tag{12}$$

Proof. Suppose $\mathcal{G}$ is an $r \times c$ balanced grid based on the v-set V. Each of the $\binom{v}{2}$ pairs of elements of V occur μ times amongst the r rows and c columns of $\mathcal{G}$. Each row covers $\binom{c}{2}$ pairs, and each column $\binom{r}{2}$ pairs. Hence

$$\binom{v}{2}\mu = r\binom{c}{2} + c\binom{r}{2},$$

and the first result follows.

Now select an arbitrary fixed element x of V; suppose x occurs $k[x]$ times in $\mathcal{G}$. Consider the pairs of the form $\{x, y\}$ where $y \in V \backslash \{x\}$. If x appears in the (i,j) cell, $r-1$ such pairs arise in column j and $c-1$ in row i, for a total of $r+c-2$ pairs. So there are $(r+c-2)k[x]$ such pairs in total. But clearly there are $v-1$ pairs of the form $\{x,y\}$ for a given x, so $k[x](r+c-2) = \mu(v-1)$. Therefore $k[x]$ is constant, k say, independent of x, and $k = \frac{rc}{v}$ from (3). □

In view of the above result, we shall denote such a balanced grid by

$$BG(v, k, \mu : r \times c).$$

The trivial cases of balanced grids are similar to those we have seen before. In the case $c = v$, a balanced grid is a Youden square. It will have $r = k$; a $BG(v, k, \mu : k \times v)$ exists if and only if there is a $(v, k, \mu - k)$-$SBIBD$. The case $c = r = v$ is a Latin square, and has $k = v, \mu = 2v$.

Theorem 4.2 *Any balanced grid $BG(v, k, \mu : r \times c)$ satisfies $v \leq r + c - 1$.*

Proof. Suppose $\mathcal{G}$ is a $BG(v, k, \mu : r \times c)$. We define two $(0, 1)$-matrices: R is the $r \times v$ matrix with $r_{ij} = 1$ if j occurs in row i of $\mathcal{G}$ and 0 otherwise. Each row of R has c entries 1 and each column has k 1's. Similarly, C is the $c \times v$ matrix with $c_{ij} = 1$ if j occurs in column i of $\mathcal{G}$ and 0 otherwise. Each row of C has r entries 1 and each column has k 1's.

Now consider the matrix $(r + c) \times v$ matrix A, defined by

$$A = \begin{bmatrix} R \\ C \end{bmatrix}.$$

$A^T = [\, R^T \; C^T \,]$ and $A^T A = [\, R^T R + C^T C \,]$. Since $\mathcal{G}$ is a balanced grid we have

$$A^T A = \begin{bmatrix} 2k & \mu & \mu & \mu & \dots & \mu \\ \mu & 2k & \mu & \mu & \dots & \mu \\ \mu & \mu & 2k & \mu & \dots & \mu \\ \vdots & \ddots & \ddots & \dots & \dots & \vdots \\ \mu & \mu & \mu & \dots & & 2k \end{bmatrix}.$$

This $v \times v$ matrix has determinant

$$(2k - \mu)^{v-1}(v\mu - \mu + 2k).$$

Suppose the determinant is zero. If $(2k - \mu) = 0$, then from Theorem 4.1 we know that $\mu = k\frac{r+c-2}{v-1}$, so $\frac{r+c-2}{v-1} = 2$ and $v = \frac{r+c}{2} \le r + c - 1$, and the Theorem is true in this case. On the other hand, if $v\mu - \mu + 2k = 0$, then $\mu = \frac{-2k}{v-1} < 0$, which is impossible.

Hence we need only consider the case where A is non-singular. So

$$v = \text{rank}(A^T A) \le \text{rank}(A) \le r + c.$$

We need to eliminate the possibility $v = r + c$.

If $v = r + c$ then from (11) and (12),

$$\mu = \frac{rc(r+c-2)}{(r+c)(r+c-1)} \text{ and } k = \frac{rc}{(r+c)},$$

and both these quantities are integers.

Now the greatest common divisor of $(r + c - 1)$ and $(r + c - 2)$ is 1, so the first equality tells us that $(r + c - 1)$ must divide rc. As $(r + c - 1)$ and $(r + c)$ are coprime, and each divides rc, so their product divides rc. But

$$\begin{aligned} r^2 + c^2 > r + c &\Rightarrow r^2 + c^2 - r - c > 0 \\ &\Rightarrow r^2 + c^2 - r - c + 2rc > 2rc \\ &\Rightarrow (r+c)(r+c-1) > 2rc > rc, \end{aligned}$$

a contradiction. □

(The above proof fails in the trivial case $r = c = 1$, but even there the Theorem is true.)

5 Triple arrays with $\mathbf{v = r + c - 1}$.

In the case $v = r + c - 1$, all parameters of a triple array can be determined from v, r and λ_{cc}, using Theorems 2.2 and 3.1. Agrawal [1] gave a method that started from a $(v + 1, r, \lambda_{cc})$-$SBIBD$ and constructed a $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$. He could not prove that it always worked, but found it to provide the required array in every case he tried, provided $r - \lambda_{cc} > 2$.

Agrawal's construction can be expressed as follows. Suppose the blocks of a $(v+1, r, \lambda_{cc})$-$SBIBD$ are $B_0, B_1, \ldots, B_v$, and suppose the elements of B_0 are labeled $e_1, e_2, \ldots, e_r$. Denote the elements of the complement of B_0 by $f_1, f_2, \ldots, f_c$ (this is the correct number of elements, because $v = r + c - 1$.) Construct an array whose (i, j) entry is the $(r - \lambda_{cc})$-set $S_{ij} = \{h : e_i \notin B_h, f_j \in B_h\}$. Then the (i, j) entry of the triple array A is an element of $S_{i,j}$, and the rows and the columns of A contain no repetitions.

Subsequently Raghavarao and Nageswarerao [12] claimed to prove that the method always works, but we have pointed out [20] that their proof is faulty. So we have

Conjecture [1, 12] *If there is a* $(v+1, r, \lambda_{cc})$*-*$SBIBD$ *with* $r - \lambda_{cc} > 2$ *then there is a* $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ *with* $v = r + c - 1$.

We shall now prove the converse of this Conjecture: the existence of the triple array implies the existence of the symmetric balanced incomplete block design.

Lemma 5.1 *Suppose* $\mathcal{A}$ *is a* $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ *with* $v = r + c - 1$. *Then*

$$\lambda_{cc} = r - \lambda_{rc} = v - 2c + \lambda_{rr} + 1. \tag{13}$$

Proof. From the data and (3) we have

$$rc = vk = (r + c - 1)k = rk + (c - 1)k,$$

so

$$r(c - 1) = r(k - 1) + (c - 1)k,$$

and, using (5),

$$r = \frac{r(k-1)}{c-1} + k = \lambda_{cc} + \lambda_{rc},$$

giving the first equality.

For the second equality, notice that

$$\lambda_{cc} - \lambda_{rr} \quad = \quad (r - \lambda_{rc}) - \frac{c(k-1)}{r-1} \text{ (from (4))}$$

$$
\begin{aligned}
&= (r-k) - \frac{c(k-1)}{r-1} \\
&= \frac{(r-k)(r-1) - c(k-1)}{r-1} \\
&= \frac{r(r-1) - k(r+c-1) + c}{r-1} \\
&= \frac{r(r-1) - kv + c}{r-1} \\
&= \frac{r(r-1) - rc + c}{r-1} \text{ (from (3))} \\
&= r - c \ = \ r + c - 1 - 2c + 1 = v - 2c + 1.
\end{aligned}
$$

□

Notice that the first equality can be written as $k = r - \lambda_{cc}$, in view of the fact that $\lambda_{rc} = k$.

Theorem 5.2 *Suppose $\mathcal{A}$ is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$. Then there exists a symmetric balanced incomplete block design with parameters*

$$(v+1, r, \lambda_{cc}). \tag{14}$$

Proof. Label the rows and columns of $\mathcal{A}$ as $R_1, R_2, \ldots, R_r$ and $C_1, C_2, \ldots, C_c$ respectively. Let $\mathcal{P}$ denote the set $\{R_1, R_2, \ldots, R_r, C_1, C_2, \ldots, C_c\}$. Then $|\mathcal{P}| = r + c = v + 1$. We construct a design with treatment set $\mathcal{P}$.

The i-th block of the $BIBD_C$, $\overline{B}_i$ say, consists of those C_j that contain entry i. We extend $\overline{B}_i$ to a block B_i as follows:

$$B_i = \overline{B}_i \cup \{R_j : i \notin R_j\}.$$

Since i belongs to k rows, it is missing from $r - k$ rows, so $|B_i| = k + (r - k) = r$. So we have v blocks of size r. We add one further block, $B_0 = \{R_1, R_2, \ldots, R_r\}$.

The pair C_iC_j occurs λ_{cc} times among the blocks $\overline{B}_i$ so it occurs λ_{cc} times among the blocks B_i. The pair C_jR_t occurs in B_i if and only if i is in column j and not in row t. Since column j intersects row t in λ_{rc} elements, it follows that C_jR_t occurs $r - \lambda_{rc}$ times; by Lemma 5.1, that is λ_{cc} times. The pair R_jR_t occurs in B_i if and only if i does not occur in row j and does not occur in row t. Since those rows intersect in λ_{rr} places, we see that R_jR_t occurs in $v-2c+\lambda_{rr}$ of the B_i. It also occurs in B_0. So it occurs in $\lambda_{cc} = v-2c+\lambda_{rr}+1$ blocks (using Lemma 5.1 again). So the design is balanced, with the required parameters. $\square$

6 Relations between the arrays

It will be convenient to denote the symmetric design (14) by $\mathcal{B}$, and to let $\mathcal{A}$ be a triple array with $v = r + c - 1$ that gives rise to $\mathcal{B}$ as described in Theorem 5.2.

Working back through the proof of Theorem 5.2, we observe that the $BIBD_C$ of the triple array is the residual design of $\mathcal{B}$ modulo the block B_0, and the $BIBD_R$ is the complement of the derived design modulo B_0.

Two symbols x and y both occur in column j of $\mathcal{A}$ if and only if blocks $\overline{B}_x$ and $\overline{B}_y$ of the $BIBD_C$ contain the common element j. So x and y will occur together in $|\overline{B}_x \cap \overline{B}_y|$ columns. Now $\overline{B}_x \cap \overline{B}_y$ consists of those elements of $B_x \cap B_y$ that do not belong to B_0, so

$$\begin{aligned} |\overline{B}_x \cap \overline{B}_y| &= |B_x \cap B_y| - |B_x \cap B_y \cap B_0| \\ &= \lambda_{cc} - |B_x \cap B_y \cap B_0|. \end{aligned} \tag{15}$$

Symbols x and y both occur in row i of $\mathcal{A}$ if and only if blocks x and y of the $BIBD_R$ contain the common element i. Those blocks are $B_0 \backslash B_x$ and $B_0 \backslash B_y$, and

$$\begin{aligned} |(B_0 \backslash B_x) \cap (B_0 \backslash B_y)| &= |B_0| - |B_0 \cap B_x| - |B_0 \cap B_y| + |B_x \cap B_y \cap B_0| \\ &= r - 2\lambda_{cc} + |B_x \cap B_y \cap B_0|. \end{aligned} \tag{16}$$

Combining (15) and (16), we see that x and y occur together $r - \lambda_{cc} = k$ times in the rows and columns of $\mathcal{A}$. We have proven:

Theorem 6.1 *Any* $TA(r+c-1, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ *is a* $BG(r+c-1, k, k : r \times c)$.

The converse is certainly not true; we have found (computationally) large numbers of balanced grids with $v = r + c - 1$ that are not triple arrays, even at size 5×6. However, we have not found a case where the balanced grid is a double array but not a triple array. So we ask:

If $\mathcal{A}$ is a double array and a balanced grid, is it necessarily a triple array? Is this true in the particular case $v = r + c - 1$?

If a balanced grid satisfies $v = r + c - 1$, then (11) and (12) give $\mu = k$.

7 The existence of balanced incomplete block designs

Although many constructions for balanced incomplete block designs are known, the existence of a design with given parameters is nearly always undecided. Apart from the necessary conditions (1) and (2), the only general nonexistence result is the Bruck-Ryser-Chowla Theorem:

Theorem 7.1 [4, 5, 13] *If there exists a symmetric balanced incomplete block design with parameters (v, k, λ), then:*

(i) if v is even, $k - \lambda$ must be a perfect square;

(ii) if v is odd, there must exist integers x, y, and z, not all zero, such that

$$x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2.$$

There is an extension to some residual designs:

Theorem 7.2 [6, 14, 15] *Suppose D is a balanced incomplete block design with parameters $(v - k, v - 1, k, k - \lambda, \lambda)$.*

(i) If $\lambda = 1$ or $\lambda = 2$, there is a (v, k, λ)-design of which D is the residual.

(ii) There is a number $f(\lambda)$, depending only on λ, such that if $k \geq f(\lambda)$, there is a (v, k, λ)-design of which D is the residual.

(The number $f(\lambda)$ grows rapidly with λ. The original bound for $f(\lambda)$, found in [14, 15], has been improved (see [8, 7]), but the best known bound is still quartic.)

A balanced incomplete block design is called *quasi-residual* if its parameters are the parameters of a residual design but the design is not the residual of any symmetric balanced incomplete block design. So Theorem 7.2 gives a restriction on the existence of quasi-residual designs.

Theorems 5.2 and 7.1 combine to show that certain triple arrays are impossible. For example, there is no $(22, 7, 2)$-$SBIBD$, so there can be no $TA(21, 5, 10,\ 2, 5 : 7 \times 15)$. Theorem 7.2 can also be used to show that some double arrays are impossible. For example, the nonexistence of a symmetric $(22, 7, 2)$-$SBIBD$ implies that there is no $(15, 21, 7, 5, 2)$-$BIBD$; so a $DA(21, 5, 10, 2 : 7 \times 15)$ is impossible (the impossible parameters are those required for its $BIBD_C$).

We were interested in parameters such that a $BIBD_R$ and a $BIBD_C$ are feasible, but the corresponding symmetric balanced incomplete block design does not exist. Such parameters may well be numerous, but few are known. Most known quasi-residual designs have the same parameters as known residual designs. For example, the first reported quasi-residual design, found by Bhattacharya [3], has parameters $(16, 24, 9, 6, 3)$. Although it cannot be embedded in a $(25, 9, 3)$-$SBIBD$, such $SBIBD$s exist. Among quasi-residual designs, many have the parameters of the residual of the complement of a non-existent projective plane, and consequently the corresponding $BIBD_R$ is impossible.

One possible candidate was discovered by van Lint, Tonchev and Landgev [18]. It has parameters $(28, 42, 15, 10, 5)$, the residual parameters of the impossible $(43, 15, 5)$-$SBIBD$. If it is the $BIBD_C$ of a double array, then the $BIBD_R$ would be a $(15, 42, 28, 10, 18)$-$BIBD$, which also exists. We very quickly found a $DA(42, 10, 18, 5 : 15 \times 28)$, which is exhibited in Appendix A.

Another interesting question is this: does there exist a double array that cannot be transformed into a triple array by any permutations of the elements within its columns, even though a triple array with the desired parameters exists? The answer is in the affirmative. We have discovered a double array $DA(24, 6, 10, 3 : 9 \times 16)$ whose $BIBD_C$ is the $(16, 24, 9, 6, 3)$-$BIBD$ given by Bhattacharya [3], who proved that this design could not be embedded in any $(25, 9, 3)$-$SBIBD$. The double array is also shown in Appendix A. (In fact, many double arrays can be constructed from Bhattacharya's design). The non-embeddability of the $BIBD_C$ means that this cannot be made into a triple array by any permutations of the elements within the columns, but a $TA(24, 6, 10, 3, 6 : 9 \times 16)$ is exhibited by Agrawal [1].

8 Triple arrays balanced for intersection

In any triple array, the rc sets formed by intersecting the supports of rows with columns are all sets of size k, and each element occurs in k^2 of them. One might ask whether these sets might ever form a balanced incomplete block design. The triple array $TA(r, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ (7) actually has this property, and its intersections form a $(10, 30, 9, 3, 2)$-$BIBD$. But this is the only case so far discovered.

If there is a triple array whose row-column intersections are balanced, then those intersections form a $(v, rc, k^2, k, \lambda')$-$BIBD$, where from (2)

$$\lambda' = k^2(k-1)/(v-1).$$

The requirement that λ' be an integer turns out to be very restrictive. A search up to $r = 100$ found only 16 sets of parameters satisfying (1) and (1) for which λ' was integral, and in 11 of these cases the existence of the corresponding symmetric balanced incomplete block design, guaranteed by Theorem 5.2, is impossible because of Theorem 7.1. (For example, the parameters work for a $TA(21, 5, 10, 2, 5 : 7 \times 15)$, but there is no $(22, 7, 2)$-$SBIBD$, so no such triple array exists.)

The smallest symmetric designs that could give row-column intersection balanced triple arrays are $(11, 5, 2)$, which gives the design (7), $(56, 11, 2)$ and $(66, 26, 10)$. In the other two cases with $r \leq 100$, namely $(149, 37, 9)$ and $(569, 72, 9)$, no symmetric balanced incomplete block design is known. We have constructed a triple array from a $(56, 11, 2)$-$SBIBD$, but it does not have the row-column intersection balance property.

9 Trivial cases of triple arrays

In this section we discuss triple arrays whose existence is a direct consequence of their parameters. These trivial cases include triple arrays with $k = 1$, $r = 1$ and $k = r$.

If $k = 1$, then every element occurs exactly once in the triple array, so $v = rc$. Distinct rows or columns will have no common members, and any row and column have intersection size 1. So the array is a $TA(rc, 1, 0, 0, 1 : r \times c)$. Such a triple array exists for all r and c.

If $r = 1$ then the array consists of a single row. Clearly $v = c$ and $k = 1$; this case is a subcase of the one discussed in the preceding paragraph.

When $k = r$, every symbol in any column must also belong to any row. So every symbol occurs in every row, and $c = v$. The array, a $TA(v, k, v, \lambda, k : k \times v)$, will be a Youden square, and as noted previously it will exist if and only if a (v, k, λ)-$SBIBD$ exists.

10 Double and triple arrays with $\mathbf{v > r + c - 1}$

Until recently, many researchers thought that there could be no triple array with $v > r + c - 1$. However, we have constructed a $TA(35, 3, 5, 1, 3 : 7 \times 15)$ (the array is presented in Appendix B). This example is very important, as it fills the gap in Table 1 of [11].

In Table 4, below, we list the parameters of possible small triple arrays with $v > r + c - 1$.

In each case, one can of course ask whether a double array exists. In particular, we found a $DA(63, 5, 6, 3 : 15 \times 21)$ and a $DA(99, 5, 18, 1 : 11 \times 45)$, which are shown in Appendix A.

11 Small arrays

Double and triple arrays and balanced grids with row or column size 1 are all trivial. If $r = 2$, the equireplicate property means that either $v = c$, a trivial case, or $v = 2c$ and $k = 1$, again trivial (and similarly, of course, for $c = 2$). It is easy to see that no balanced grid with $r = 2$ can exist.

The case $r = 3$ is interesting. The equation $vk = rc$ and the non-triviality condition $v > c$ imply $k < 3$, so the only non-trivial case is $k = 2$. For a double or triple array, equation (5) yields $c = 4$, so $v = 6$. There is a double array $DA(6, 2, 2, 1 : 3 \times 4)$, for example

$$\begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 4 & 1 & 5 & 6 \\ 5 & 3 & 6 & 2 \\ \hline \end{array},$$

but no $TA(6, 2, 2, 1, 2 : 3 \times 4)$ exists. If a non-trivial balanced grid with $r = 3$ exists, then equations (11) and (12) imply that $\mu = \frac{4c+4}{3c-2}$; the only solution is $c = 4, \mu = 2$ and it may be shown by exhaustion that no such grid exists.

We now discuss small non-trivial double and triple arrays.

We saw in Theorem 5.2 that a triple array with $v = r + c - 1$ exists only if a certain symmetric balanced incomplete block design exists. In Table 1 we list the parameters of triple arrays of this kind with $r \leq 16$. The notation A means that the triple array was constructed by Agrawal and appears in [1]. However, it should be noted that a $TA(10, 3, 3, 2, 3 : 5 \times 6)$ may be found in [9] and a $TA(14, 4, 4, 3, 4 : 7 \times 8)$ is constructed in [10]. We found it very easy to implement Agrawal's method by computer, and C means that a triple array with the required parameters was constructed by us. Many of these triple arrays can be found in Appendix B; for reasons of space, larger arrays have been omitted, but they are listed at [21]. Preece [11] points out that a $TA(22, 6, 6, 5, 6 : 11 \times 12)$ can be constructed by omitting a factor from a design obtainable by Method 2.6 of [2], but our example was found by computer search.

Table 2 shows the cases (for $r \leq 16$) where the existence of a symmetric design is undecided.

Table 3 lists the parameters of double arrays for those cases with $v = r + c - 1$ ($r \leq 16$) in which a triple array does not exist, the corresponding symmetric design being impossible. In many cases it is also known (from Theorem 7.2) that a $BIBD_C$ does not exist, so no DA is possible; but not always. There is also the one anomalous case, corresponding to the (unique) $(7, 3, 1)$-$SBIBD$. Table 4 is a list of parameters with $v > r + c - 1$, up to

TA	$BIBD_R$	$BIBD_C$	$SBIBD$	
$v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$	r, v, c, k, λ_{rr}	c, v, r, k, λ_{cc}	$v+1, r, \lambda_{cc}$	
$12, 3, 6, 1, 3 : 4 \times 9$	$4, 12, 9, 3, 6$	$9, 12, 4, 3, 1$	$13, 4, 1$	A
$10, 3, 3, 2, 3 : 5 \times 6$	$5, 10, 6, 3, 3$	$6, 10, 5, 3, 2$	$11, 5, 2$	A
$20, 4, 12, 1, 4 : 5 \times 16$	$5, 20, 16, 4, 12$	$16, 20, 5, 4, 1$	$21, 5, 1$	A
$15, 4, 6, 2, 4 : 6 \times 10$	$6, 15, 10, 4, 6$	$10, 15, 6, 4, 2$	$16, 6, 2$	A
$30, 5, 20, 1, 5 : 6 \times 25$	$6, 30, 25, 5, 20$	$25, 30, 6, 5, 1$	$31, 6, 1$	A
$14, 4, 4, 3, 4 : 7 \times 8$	$7, 14, 8, 4, 4$	$8, 14, 7, 4, 3$	$15, 7, 3$	A
$56, 7, 42, 1, 7 : 8 \times 49$	$8, 56, 49, 7, 42$	$49, 56, 8, 7, 1$	$57, 8, 1$	C
$18, 5, 5, 4, 5 : 9 \times 10$	$9, 18, 10, 5, 5$	$10, 18, 9, 5, 4$	$19, 9, 4$	A
$24, 6, 10, 3, 6 : 9 \times 16$	$9, 24, 16, 6, 10$	$16, 24, 9, 6, 3$	$25, 9, 3$	A
$36, 7, 21, 2, 7 : 9 \times 28$	$9, 36, 28, 7, 21$	$28, 36, 9, 7, 2$	$37, 9, 2$	C
$72, 8, 56, 1, 8 : 9 \times 64$	$9, 72, 64, 8, 56$	$64, 72, 9, 8, 1$	$73, 9, 1$	C
$30, 7, 14, 3, 7 : 10 \times 21$	$10, 30, 21, 7, 14$	$21, 30, 10, 7, 3$	$31, 10, 3$	C
$90, 9, 72, 1, 9 : 10 \times 81$	$10, 90, 81, 9, 72$	$81, 90, 10, 9, 1$	$91, 10, 1$	C
$22, 6, 6, 5, 6 : 11 \times 12$	$11, 22, 12, 6, 6$	$12, 22, 11, 6, 5$	$23, 11, 5$	C
$55, 9, 36, 2, 9 : 11 \times 45$	$11, 55, 45, 9, 36$	$45, 55, 11, 9, 2$	$56, 11, 2$	C
$44, 9, 24, 3, 9 : 12 \times 33$	$12, 44, 33, 9, 24$	$33, 44, 12, 9, 3$	$45, 12, 3$	C
$132, 11, 110, 1, 11 : 12 \times 121$	$12, 132, 121, 11, 110$	$121, 132, 12, 11, 1$	$133, 12, 1$	C
$26, 7, 7, 6, 7 : 13 \times 14$	$13, 26, 14, 7, 7$	$14, 26, 13, 7, 6$	$27, 13, 6$	C
$39, 9, 18, 4, 9 : 13 \times 27$	$13, 39, 27, 9, 18$	$27, 39, 13, 9, 4$	$40, 13, 4$	C
$78, 11, 55, 2, 11 : 13 \times 66$	$13, 78, 66, 11, 55$	$66, 78, 13, 11, 2$	$79, 13, 2$	C
$182, 13, 156, 1, 13 : 14 \times 169$	$14, 182, 169, 13, 156$	$169, 182, 14, 13, 1$	$183, 14, 1$	C
$30, 8, 8, 7, 8 : 15 \times 16$	$15, 30, 16, 8, 8$	$16, 30, 15, 8, 7$	$31, 15, 7$	C
$35, 9, 12, 6, 9 : 15 \times 21$	$15, 35, 21, 9, 12$	$21, 35, 15, 9, 6$	$36, 15, 6$	C
$70, 12, 44, 3, 12 : 15 \times 56$	$15, 70, 56, 12, 44$	$56, 70, 15, 12, 3$	$71, 15, 3$	C
$40, 10, 15, 6, 10 : 16 \times 25$	$16, 40, 25, 10, 15$	$25, 40, 16, 10, 6$	$41, 16, 6$	C
$48, 11, 22, 5, 11 : 16 \times 33$	$16, 48, 33, 11, 22$	$33, 48, 16, 11, 5$	$49, 16, 5$	C
$60, 12, 33, 4, 12 : 16 \times 45$	$16, 60, 45, 12, 33$	$45, 60, 16, 12, 4$	$61, 16, 4$	C

Table 1: Parameters for a TA with $v = r + c - 1$ where the $SBIBD$ is known.
$2 \leq r \leq 16, r \leq c, 2 \leq k < r, 1 \leq \lambda_{cc} < r$.

$r = 16$, for a possible double or triple array. In both these tables C again denotes an array we have constructed, while NE means that the indicated array is non-existent.

TA	$BIBD_R$	$BIBD_C$	$SBIBD$
$v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$	r, v, c, k, λ_{rr}	c, v, r, k, λ_{cc}	$v + 1, r, \lambda_{cc}$
$156, 12, 132, 1, 12 : 13 \times 144$	$13, 156, 144, 12, 132$	$144, 156, 13, 12, 1$	$157, 13, 1$
$80, 13, 52, 3, 13 : 16 \times 65$	$16, 80, 65, 13, 52$	$65, 80, 16, 13, 3$	$81, 16, 3$
$120, 14, 91, 2, 14 : 16 \times 105$	$16, 120, 105, 14, 91$	$105, 120, 16, 14, 2$	$121, 16, 2$
$240, 15, 210, 1, 15 : 16 \times 225$	$16, 240, 225, 15, 210$	$225, 240, 16, 15, 1$	$241, 16, 1$

Table 2: Case $v = r + c - 1$. Possible parameters for a TA when no $SBIBD$ is known. $2 \leq r \leq 16, r \leq c, 2 \leq k < r, 1 \leq \lambda_{cc} < r$.

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TA	$BIBD_R$	$BIBD_C$	$SBIBD$	DA ?
$v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$	r, v, c, k, λ_{rr}	c, v, r, k, λ_{cc}	$v+1, r, \lambda_{cc}$	
$6, 2, 2, 1, 2 : 3 \times 4$	$3, 6, 4, 2, 2$	$4, 6, 3, 2, 1$	$7, 3, 1$	C
$21, 5, 10, 2, 5 : 7 \times 15$	$7, 21, 15, 5, 10$	$15, 21, 7, 5, 2$	$22, 7, 2$	NE
$42, 6, 30, 1, 6 : 7 \times 36$	$7, 42, 36, 6, 30$	$36, 42, 7, 6, 1$	$43, 7, 1$	NE
$28, 6, 15, 2, 6 : 8 \times 21$	$8, 28, 21, 6, 15$	$21, 28, 8, 6, 2$	$29, 8, 2$	NE
$45, 8, 28, 2, 8 : 10 \times 36$	$10, 45, 36, 8, 28$	$36, 45, 10, 8, 2$	$46, 10, 2$	NE
$110, 10, 90, 1, 10 : 11 \times 100$	$11, 110, 100, 10, 90$	$100, 110, 11, 10, 1$	$111, 11, 1$	NE
$33, 8, 14, 4, 8 : 12 \times 22$	$12, 33, 22, 8, 14$	$22, 33, 12, 8, 4$	$34, 12, 4$	??
$66, 10, 45, 2, 10 : 12 \times 55$	$12, 66, 55, 10, 45$	$55, 66, 12, 10, 2$	$67, 12, 2$	NE
$52, 10, 30, 3, 10 : 13 \times 40$	$13, 52, 40, 10, 30$	$40, 52, 13, 10, 3$	$53, 13, 3$	??
$91, 12, 66, 2, 12 : 14 \times 78$	$14, 91, 78, 12, 66$	$78, 91, 14, 12, 2$	$92, 14, 2$	NE
$42, 10, 18, 5, 10 : 15 \times 28$	$15, 42, 28, 10, 18$	$28, 42, 15, 10, 5$	$43, 15, 5$	C
$105, 13, 78, 2, 13 : 15 \times 91$	$15, 105, 91, 13, 78$	$91, 105, 15, 13, 2$	$106, 15, 2$	NE
$210, 14, 182, 1, 14 : 15 \times 196$	$15, 210, 196, 14, 182$	$196, 210, 15, 14, 1$	$211, 15, 1$	NE

Table 3: Case $v = r + c - 1$. Parameters for which there is no TA.
$2 \leq r \leq 16, r \leq c, 2 \leq k < r, 1 \leq \lambda_{cc} < r$.

TA	$BIBD_R$	$BIBD_C$	DA ?
$v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$	r, v, c, k, λ_{rr}	c, v, r, k, λ_{cc}	TA ?
$35, 3, 5, 1, 3 : 7 \times 15$	$7, 35, 15, 3, 5$	$15, 35, 7, 3, 1$	TA C
$99, 5, 18, 1, 5 : 11 \times 45$	$11, 99, 45, 5, 18$	$45, 99, 11, 5, 1$	DA C
$130, 4, 10, 1, 4 : 13 \times 40$	$13, 130, 40, 4, 10$	$40, 130, 13, 4, 1$	??
$63, 5, 6, 3, 5 : 15 \times 21$	$15, 63, 21, 5, 6$	$21, 63, 15, 5, 3$	DA C
$195, 7, 39, 1, 7 : 15 \times 91$	$15, 195, 91, 7, 39$	$91, 195, 15, 7, 1$	??
$100, 4, 5, 2, 4 : 16 \times 25$	$16, 100, 25, 4, 5$	$25, 100, 16, 4, 2$	??
$216, 6, 27, 1, 6 : 16 \times 81$	$16, 216, 81, 6, 27$	$81, 216, 16, 6, 1$	??
$56, 6, 7, 4, 6 : 16 \times 21$	$16, 56, 21, 6, 7$	$21, 56, 16, 6, 4$	??
$232, 10, 87, 1, 10 : 16 \times 145$	$16, 232, 145, 10, 87$	$145, 232, 16, 10, 1$	??

Table 4: Case $v > r + c - 1$. Possible parameters for a DA or TA.
$2 \leq r \leq 16, r \leq c, 2 \leq k < r, 1 \leq \lambda_{cc} < r$.

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[21] Triple arrays, .

Appendix A: Double arrays

This is a double array $DA(6, 2, 2, 1 : 3 \times 4)$:

1	2	3	4
4	1	5	6
5	3	6	2

Here is the $DA(42, 10, 18, 5 : 15 \times 28)$, mentioned in Section 7:

16	31	26	27	7	21	2	12	13	35	37	20	39	30	8	3	34	24	5	40	9	18	10	32	33	22	23	38
40	38	22	21	13	25	31	27	14	11	34	41	28	32	4	9	3	23	8	6	19	10	33	36	24	39	17	35
34	12	20	9	42	14	41	39	4	15	26	1	36	18	32	5	10	40	37	24	28	35	11	23	22	33	7	25
37	25	36	12	6	1	35	24	10	42	19	29	8	34	5	38	2	11	16	27	33	26	21	15	13	23	41	40
9	6	11	34	35	29	24	25	28	27	13	7	30	41	2	22	26	3	12	38	17	39	36	16	42	14	20	37
31	35	7	14	37	40	38	1	25	3	15	28	21	39	30	36	23	27	4	10	8	29	13	17	12	18	26	42
11	18	19	39	15	8	22	16	24	32	1	13	31	4	37	26	27	28	41	30	38	9	29	40	5	2	14	36
2	3	31	19	32	10	12	9	41	37	40	14	42	1	23	25	20	5	28	16	27	33	30	38	17	6	15	39
6	10	39	41	28	11	26	13	17	29	4	2	3	16	42	1	38	21	33	32	7	40	18	20	34	15	24	31
3	7	8	32	21	34	15	42	39	18	12	5	2	22	35	40	33	4	25	14	11	17	27	29	30	16	19	41
20	17	42	5	22	31	9	23	34	12	28	35	29	3	1	19	4	30	6	26	16	8	41	13	18	40	36	33
5	41	18	6	24	7	23	10	2	14	31	27	15	9	17	32	42	35	29	36	34	20	19	4	21	13	37	30
18	19	5	20	10	22	30	36	11	25	6	31	14	7	16	28	29	33	24	8	42	1	37	21	38	35	3	32
17	21	4	23	20	37	8	31	32	26	30	38	6	29	19	2	22	39	36	7	15	12	9	11	1	25	33	34
24	4	13	8	9	23	11	3	26	5	27	16	17	15	10	21	6	7	22	18	25	28	2	12	14	19	1	20

The $DA(24, 6, 10, 3 : 9{\times}16)$ whose column design is the $(16, 24, 9, 6, 3)$-$BIBD$ due to Bhattacharya [3] (see Section 7):

21	18	11	19	23	12	7	1	3	6	20	14	5	4	9	13
23	3	4	12	2	5	11	13	21	16	24	17	6	8	15	20
17	10	12	9	6	22	1	16	4	14	11	20	8	23	24	19
16	1	3	10	12	18	13	14	9	7	2	4	24	11	17	15
1	12	8	13	4	15	5	2	23	21	10	7	19	14	22	24
13	6	2	14	20	11	18	22	10	15	17	5	16	19	1	21
5	17	21	7	19	16	2	4	22	11	23	9	10	18	8	3
19	20	17	22	15	8	9	3	18	23	13	24	2	1	6	7
12	14	7	8	9	20	6	15	5	10	22	16	3	24	21	18

A $DA(63, 5, 6, 3 : 15 \times 21)$:

24	15	37	36	58	50	13	55	14	44	16	46	61	27	42	29	1	31	51	10	23
45	28	14	52	11	59	51	56	1	2	17	38	15	37	62	43	30	24	32	25	47
48	57	29	15	16	12	60	31	53	33	1	44	39	25	38	18	2	3	63	52	26
54	34	26	27	39	61	30	19	32	58	13	17	45	4	49	1	40	2	3	46	53
14	55	16	59	31	40	33	35	62	54	3	18	47	20	41	27	28	50	4	5	2
3	29	5	6	32	17	55	21	15	63	34	56	36	19	28	60	51	41	42	4	48
7	49	56	30	52	33	18	42	35	57	37	4	5	61	20	22	43	29	46	1	6
6	50	58	38	7	53	62	34	57	36	43	2	19	16	21	8	23	44	30	47	5
35	7	51	9	37	8	39	63	20	16	58	59	3	48	17	6	22	45	24	54	44
49	4	8	17	18	38	9	52	36	23	59	25	60	40	7	21	61	10	45	31	55
56	8	50	31	53	19	41	10	22	37	24	60	18	46	5	39	9	62	11	26	32
12	46	9	51	10	32	54	40	11	42	38	23	57	6	47	63	19	20	25	33	27
33	13	47	10	28	11	20	61	41	12	55	39	24	58	26	48	7	52	21	43	34
27	25	35	48	49	29	34	13	56	21	22	14	40	62	59	42	44	8	53	12	11
28	36	30	57	60	54	12	14	43	15	45	35	26	41	63	50	49	23	9	22	13

A $DA(99, 5, 18, 1 : 11 \times 45)$:

34	12	86	91	15	1	75	71	53	92	64	56	81	93	59	20	5	31	82	3	26	27	36
81	82	83	48	4	61	92	21	37	90	15	2	93	79	10	59	27	17	57	94	16	50	28
50	72	91	84	49	11	62	36	77	5	29	93	61	82	14	51	60	88	94	58	95	66	6
19	7	8	17	18	39	6	92	89	23	84	67	41	4	28	26	94	1	29	83	70	40	78
71	35	13	9	85	79	2	52	8	73	93	7	68	42	5	84	16	62	63	90	49	57	46
91	20	31	14	10	86	25	3	64	9	6	85	8	69	63	94	61	28	32	30	76	17	58
11	70	66	87	75	92	51	76	4	65	44	77	26	62	43	86	22	53	18	64	9	21	95
5	1	16	32	33	74	87	88	22	38	60	30	78	27	93	65	52	56	54	19	89	10	67
6	46	2	67	88	24	12	13	72	78	1	45	57	9	83	34	35	94	89	55	39	90	86
89	91	73	3	68	10	40	7	14	54	76	65	12	58	80	21	87	23	2	25	56	95	18
69	90	47	74	91	55	80	63	92	15	66	11	3	13	70	4	85	81	24	33	25	77	22

47	48	14	80	23	42	78	4	97	16	69	60	9	25	37	49	58	45	70	38	67	89
87	60	54	71	76	24	13	35	5	43	72	70	46	65	6	38	98	49	32	26	39	68
7	38	16	22	18	28	25	80	55	71	17	3	40	47	73	83	39	69	99	33	27	44
59	95	96	15	72	12	50	74	37	51	52	73	85	81	56	62	45	61	30	63	34	48
95	24	30	96	27	19	29	41	20	38	97	18	31	60	82	74	75	51	86	53	64	40
19	80	75	47	83	96	74	54	42	97	39	53	41	72	21	98	50	36	52	87	43	65
37	88	81	55	48	73	20	97	31	32	59	40	98	10	33	7	84	29	15	42	54	99
23	8	21	82	96	49	85	66	76	77	44	45	71	98	43	34	63	99	41	11	12	55
79	20	44	17	11	77	53	56	75	68	33	97	64	42	61	22	23	31	50	66	99	28
29	69	79	45	51	84	43	36	67	1	78	34	24	32	98	57	35	90	62	46	47	13
68	30	46	26	41	52	96	19	57	58	2	79	59	36	48	44	8	14	37	99	88	35

Appendix B: Triple arrays

Here is a $TA(35,3,5,1,3 : 7 \times 15)$; see Section 10:

31	1	18	16	7	10	5	3	4	2	33	14	19	15	12
26	32	1	2	29	30	28	20	27	11	5	34	3	8	4
1	17	13	9	3	4	21	22	6	35	25	5	24	2	23
6	27	33	28	16	13	35	30	15	10	9	26	12	17	29
16	12	23	32	34	21	15	33	24	22	11	10	8	25	20
21	22	28	24	25	19	7	14	18	29	27	23	26	30	31
11	7	8	14	13	32	20	6	34	18	19	17	35	31	9

.

The following arrays were constructed by computer, using Agrawal's construction. They fill in the holes in Table 1 that are marked C.

A $TA(56,7,42,1,7 : 8 \times 49)$

50	23	31	46	26	20	35	33	34	19	37	38	47	51	39	49	21	24	14	52	27	11	15	16	40
36	50	3	39	40	41	49	51	18	44	20	22	15	24	1	52	43	4	34	46	29	37	2	26	53
29	30	45	25	12	50	7	9	26	27	4	46	51	40	52	31	3	44	36	37	38	47	53	49	43
8	44	17	50	47	13	42	41	51	11	29	21	14	32	48	40	12	52	16	6	7	34	35	53	30
22	9	10	4	5	48	50	49	43	51	45	13	6	8	28	2	23	42	52	26	47	21	25	39	14
43	2	24	18	50	27	28	1	10	35	51	30	31	7	19	22	32	13	45	17	52	53	48	3	4
15	37	50	11	19	34	21	25	2	3	12	51	39	16	30	20	52	33	5	35	9	1	38	29	17
1	16	38	32	33	6	14	17	42	36	28	5	23	48	10	11	41	15	25	8	18	24	12	13	27

53	32	36	42	28	54	8	43	25	41	13	29	30	48	10	55	12	44	45	9	22	17	18	56
5	45	7	16	32	48	54	35	21	30	23	55	25	31	27	33	17	38	56	28	47	42	6	19
41	42	23	1	54	33	34	39	6	11	35	14	8	55	5	28	22	32	2	56	10	48	24	13
18	9	46	31	36	3	19	20	10	54	45	2	55	4	37	38	39	1	33	15	56	5	49	43
44	19	53	12	17	18	38	54	40	15	1	24	3	36	55	16	7	20	27	46	41	11	56	37
8	6	33	54	47	14	49	5	29	26	55	46	20	9	15	11	44	56	21	34	16	23	12	25
28	53	10	27	13	22	23	24	54	7	18	41	42	26	32	6	55	14	8	40	4	56	36	31
31	22	20	46	2	37	4	9	44	45	40	19	47	21	49	43	34	26	39	3	35	29	30	7

A $TA(36,7,21,2,7:9\times 28)$:

1	8	6	11	3	30	36	16	13	9	2	7	33	5	25	17	28	23	10	27	12	24	15	20	29	18	19	31
33	18	4	29	12	31	8	7	10	35	17	15	3	9	5	19	22	20	11	25	13	27	14	26	30	1	32	21
15	30	20	33	8	6	32	34	27	11	21	2	16	18	10	26	9	22	23	4	5	14	19	28	36	31	12	13
36	1	10	7	30	23	14	33	11	12	15	18	8	4	16	20	13	25	24	22	17	6	28	29	21	32	35	34
27	4	1	5	22	35	31	12	34	36	14	13	23	20	19	7	21	10	26	24	28	18	30	16	17	29	2	3
35	25	3	28	5	7	24	26	2	18	31	32	14	17	6	22	1	9	21	11	23	33	27	8	16	34	30	20
23	26	27	21	6	4	10	9	35	28	12	22	17	34	18	36	8	2	3	31	25	29	7	15	32	19	24	33
22	2	28	4	34	13	5	10	17	14	1	3	19	15	35	6	24	29	30	16	32	26	25	31	9	11	33	36
5	36	32	2	29	9	7	1	8	3	6	16	4	24	21	11	19	14	15	12	26	13	34	35	27	23	20	25

A $TA(72,8,56,1,8:9\times 64)$: — See [21]

A $TA(30,7,14,3,7:10\times 21)$:

18	19	2	29	22	11	1	16	13	30	6	3	23	24	12	10	21	8	26	7	15
19	3	23	15	16	29	17	30	4	14	24	25	1	20	2	11	7	12	13	27	9
25	14	4	24	29	16	18	21	28	5	2	8	26	30	3	17	12	13	20	10	1
29	22	8	13	6	5	26	11	30	27	17	9	21	2	19	3	18	25	14	15	4
28	29	12	3	10	7	27	6	15	18	23	30	4	26	5	20	14	19	9	8	16
10	11	29	21	24	8	9	27	7	22	30	20	19	5	17	13	4	6	16	1	28
7	20	21	9	15	25	29	28	12	23	14	18	30	10	22	6	5	1	2	17	11
13	28	20	23	25	26	24	12	17	16	19	22	9	27	10	18	11	15	21	14	8
2	26	27	5	4	17	6	3	22	1	28	24	25	15	16	23	19	20	7	21	18
27	1	22	28	14	23	12	25	26	13	8	7	10	11	9	4	24	5	6	3	2

A $TA(90,9,72,1,9:10\times 81)$: — See [21]

A $TA(22, 6, 6, 5, 6 : 11 \times 12)$:

13	21	15	7	19	10	20	4	22	1	3	5
6	16	8	4	17	22	11	13	19	20	1	2
22	4	6	15	10	9	2	21	3	18	17	16
20	9	10	14	16	18	7	8	21	2	19	1
12	5	18	19	13	11	10	14	2	6	21	22
21	20	11	16	6	13	3	1	12	15	9	14
18	17	19	9	12	1	22	15	7	11	14	4
8	12	14	6	7	16	17	10	15	5	22	20
7	11	4	13	3	5	14	17	16	21	8	18
9	19	20	2	18	15	13	12	8	17	5	3
4	10	3	5	1	8	9	2	11	7	6	12

A $TA(55, 9, 36, 2, 9 : 11 \times 45)$: — See [21]

A $TA(22, 6, 6, 5, 6 : 12 \times 33)$:

34	38	6	44	8	41	29	22	43	13	32	25	15	4	12	5	26	17	10	20	28	19	31	23	9	7	11	16	37	18	14	40	35
37	26	36	28	27	38	10	11	33	14	30	24	20	12	23	43	44	2	18	16	21	15	13	9	34	3	1	7	8	31	17	41	40
22	2	4	38	35	19	24	40	21	15	18	44	1	14	6	13	3	43	30	11	5	16	17	32	36	27	25	37	33	12	41	29	10
31	20	40	5	9	8	36	34	13	44	43	17	42	21	7	1	4	15	29	30	25	26	32	2	24	22	6	38	14	39	3	18	16
28	35	33	21	29	22	43	10	26	27	19	40	9	39	44	11	18	7	6	2	23	3	8	31	4	36	16	1	17	38	42	12	5
40	1	20	24	43	30	25	32	42	23	34	33	44	19	2	9	13	39	14	28	3	11	27	10	12	5	15	4	38	35	6	7	8
3	44	43	41	32	36	12	39	22	31	23	28	17	8	16	24	37	25	1	27	29	5	33	4	26	14	35	10	34	15	9	42	30
1	41	30	6	21	7	20	44	35	29	16	39	19	43	14	18	27	26	28	9	13	31	12	25	2	34	36	32	5	37	11	33	42
43	29	37	34	24	44	31	21	28	36	17	35	10	3	42	22	11	41	8	15	30	7	20	33	19	18	23	39	2	6	32	4	13
25	3	26	31	7	33	11	30	38	39	27	16	21	20	37	23	42	6	19	4	17	24	22	14	35	10	34	13	29	9	40	1	41
2	23	5	4	40	25	38	12	14	19	42	20	37	41	24	39	8	27	26	22	7	32	1	18	17	35	21	36	11	30	28	15	31
19	32	23	27	37	9	42	26	15	40	38	18	5	16	22	41	25	10	24	29	12	33	6	21	13	20	8	28	39	3	36	34	2

A $TA(132, 11, 110, 1, 11 : 12 \times 121)$: — See [21]

A $TA(26, 7, 7, 6, 7 : 13 \times 14)$:

6	8	23	20	7	26	12	22	15	3	14	17	21	9
25	1	10	8	5	15	9	11	24	23	17	14	19	22
26	16	9	6	18	12	24	2	23	25	10	11	14	20
17	23	21	11	25	2	18	14	4	7	15	1	3	16
12	5	16	17	19	24	26	4	22	14	8	18	2	3
5	15	7	18	26	21	4	10	14	24	6	19	20	1
11	4	6	3	9	1	10	5	2	13	12	7	8	14
15	20	2	25	4	10	17	26	7	8	16	13	9	19
19	18	3	7	23	11	13	20	12	16	5	22	15	10
20	6	17	4	11	8	23	21	19	1	13	16	24	12
2	22	5	24	12	7	1	17	25	20	18	9	13	21
1	2	19	10	8	22	3	23	13	26	21	6	18	25
3	9	24	22	21	16	15	13	6	5	25	26	11	4

A $TA(78, 11, 55, 2, 11 : 13 \times 66)$: — See [21]

A $TA(182, 13, 156, 1, 13 : 14 \times 169)$: — See [21]

A $TA(30, 8, 8, 7, 8 : 15 \times 16)$

25	26	20	9	6	14	2	3	24	30	18	27	16	29	1	15
9	5	21	23	19	1	18	27	29	2	30	12	6	28	4	17
28	6	9	22	7	26	10	23	11	24	3	1	2	17	14	4
2	3	4	18	23	8	15	29	12	11	10	5	7	24	27	25
4	9	3	12	24	11	5	28	19	16	13	26	8	25	30	6
8	10	6	11	21	15	27	14	30	18	1	28	26	5	13	7
7	17	30	29	9	12	13	16	20	3	23	15	28	8	10	1
20	19	11	1	12	16	24	13	17	21	5	7	27	14	3	2
14	8	18	6	13	25	12	17	28	15	20	3	4	21	2	22
16	21	7	5	14	4	9	18	15	23	26	22	29	3	19	13
10	15	16	24	30	23	17	6	14	8	4	19	20	27	22	5
19	29	12	8	25	17	22	10	18	1	24	6	21	7	26	16
18	11	27	20	2	9	26	25	7	19	17	23	13	22	8	30
1	20	10	13	15	27	28	19	4	22	21	25	24	9	29	11
5	2	22	14	10	21	16	11	26	28	25	20	23	30	12	29

A $TA(35, 9, 12, 6, 9 : 15 \times 21)$

31	8	28	32	23	6	19	34	20	22	9	24	7	4	33	35	30	17	12	25	15
9	29	7	8	2	1	16	32	10	4	34	13	24	12	5	30	35	33	27	14	31
13	9	8	15	26	16	14	3	24	12	11	7	2	25	19	17	18	23	22	21	28
28	7	27	35	34	29	8	30	25	10	23	5	22	16	26	6	13	20	33	32	3
7	11	21	9	20	5	17	13	34	35	2	12	18	26	1	10	23	19	4	30	16
19	4	17	22	3	32	23	14	16	28	15	33	25	27	21	26	31	29	1	5	11
34	17	31	25	4	10	6	7	1	33	13	35	29	30	27	2	5	3	32	28	19
4	6	14	19	15	2	18	5	17	16	12	20	33	3	9	21	24	22	7	8	26
22	26	9	1	18	13	28	17	32	2	30	15	34	14	25	27	21	24	20	23	33
5	20	2	18	6	35	15	16	12	17	32	14	11	8	29	25	9	10	31	33	24
25	14	24	7	12	21	27	18	11	26	10	3	23	22	16	20	19	15	29	31	9
16	35	11	12	28	3	33	26	30	18	19	25	27	24	34	31	1	4	17	2	6
8	5	1	3	31	27	35	15	18	9	4	11	6	23	14	19	20	21	13	10	29
10	23	34	29	1	24	13	21	6	11	27	28	26	31	20	15	22	8	30	18	32
6	32	3	2	5	4	1	22	8	21	14	10	13	35	12	7	11	34	28	29	30

A $TA(70, 12, 44, 3, 12 : 15 \times 56)$: — See [21]

A $TA(40, 10, 15, 6, 10 : 16 \times 25)$:

5	22	36	18	40	6	39	1	21	13	19	4	12	32	33	9	35	15	8	24	7	23	17	30	16
24	40	31	26	33	36	10	11	37	5	14	23	2	19	13	18	7	22	34	17	8	9	25	20	1
34	1	2	32	35	23	15	25	16	27	36	24	18	12	14	6	10	38	37	8	20	3	9	21	19
21	6	10	2	25	17	37	20	7	22	16	3	4	28	35	39	31	33	9	38	13	24	19	15	11
31	29	3	39	38	22	18	40	4	17	5	12	25	34	20	10	32	7	11	16	23	8	14	6	21
36	28	33	22	23	14	29	13	9	10	35	20	27	5	2	12	40	6	39	11	3	21	32	26	24
4	25	30	3	34	1	7	24	14	36	22	37	16	33	15	29	13	12	23	40	28	31	27	10	6
28	5	7	37	6	4	23	8	26	21	2	35	24	25	38	36	14	32	13	34	30	29	15	11	17
1	8	9	24	7	37	38	30	39	15	3	26	5	21	29	35	11	31	33	14	12	18	22	25	27
8	32	23	4	9	28	12	38	2	40	34	15	11	39	22	31	21	19	27	30	25	13	10	1	26
10	16	32	8	27	39	40	19	25	3	11	17	36	14	1	15	18	28	7	35	2	26	4	33	29
27	9	29	30	4	11	2	36	20	37	31	40	33	3	21	17	6	8	15	12	18	28	5	16	34
7	35	17	34	19	9	5	3	36	26	30	11	38	1	18	13	37	20	32	10	22	4	29	28	12
35	31	38	10	5	18	19	6	12	8	39	1	13	37	4	26	30	14	29	33	27	16	20	23	2
20	2	1	6	3	27	28	39	30	38	15	31	32	13	40	34	19	26	16	9	17	14	24	5	7
39	37	21	33	26	38	24	29	40	16	23	32	31	17	34	25	27	36	20	28	35	19	30	18	22

A $TA(48, 11, 22, 5, 11 : 16 \times 33)$:

37	44	41	17	4	3	39	8	25	11	6	23	38	2	22	31	14	26	43	28	34	12	21	46	18	1	29	7	32	35	16	9	42
9	35	28	8	40	21	32	15	24	44	2	12	34	6	20	11	47	45	13	29	16	25	18	48	36	30	17	42	39	3	19	22	41
10	5	18	32	38	1	23	13	19	30	29	39	8	33	26	44	25	2	42	35	12	3	36	17	7	24	27	43	47	40	9	4	22
45	14	43	6	19	41	12	1	35	40	46	30	32	21	15	5	4	8	31	39	20	22	23	13	24	17	26	29	16	44	27	48	38
35	8	36	33	46	17	10	7	21	9	13	41	11	31	37	32	15	43	24	20	38	26	30	47	44	2	4	14	28	12	25	45	19
38	25	46	27	36	9	31	10	26	41	47	42	39	34	17	40	7	19	48	12	14	23	28	18	3	21	30	45	5	4	8	6	20
41	28	27	14	3	26	1	11	39	23	37	15	6	24	35	48	18	5	4	40	25	20	44	31	8	34	19	9	13	33	43	36	2
5	40	22	46	11	15	36	9	38	31	42	4	47	20	39	37	29	12	34	48	44	21	8	33	26	10	16	13	2	18	32	14	27
23	46	2	36	15	47	48	17	11	29	19	16	4	5	7	24	21	34	9	30	45	27	37	22	31	39	25	44	14	10	3	1	35
18	42	9	35	32	28	43	6	36	47	5	14	27	23	12	17	24	29	3	33	1	11	38	45	46	37	15	34	40	31	21	13	30
29	22	5	41	7	48	28	16	33	10	17	24	45	8	36	42	40	11	20	4	23	14	35	43	1	18	34	19	26	21	37	32	47
15	10	24	48	34	22	2	19	46	16	23	33	28	7	27	12	38	37	35	9	40	29	13	8	42	26	31	41	45	6	47	25	1
32	15	45	3	22	44	42	14	6	26	28	37	10	9	43	27	48	38	17	11	2	24	16	23	20	40	33	39	7	19	46	30	18
20	30	16	2	23	5	21	12	42	43	4	47	44	32	34	7	6	3	10	46	33	13	48	25	15	41	18	27	35	45	1	26	11
39	43	4	29	41	33	15	5	16	36	20	8	1	22	6	25	2	30	38	7	17	10	27	11	45	13	28	21	3	37	40	42	31
25	34	12	10	13	38	19	18	5	6	44	3	24	30	47	41	43	48	16	1	36	28	7	39	29	46	32	33	31	22	20	37	14

A $TA(60, 12, 33, 4, 12 : 16 \times 45)$: — See [21]