A Combinatorial Interpretation of Bessel Polynomials and their First Derivatives as Ordered Hit Polynomials

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Abstract

Consider the hit polynomial of the path P_{2n} embedded in the complete graph K_{2n}. We give a combinatorial interpretation of the n-th Bessel polynomial in terms of a modification of this hit polynomial, called the ordered hit polynomial. Also, the first derivative of the n-th Bessel polynomial is shown to be the ordered hit polynomial of P_{2n-1} embedded in K_{2n}.

1 Introduction: Bessel polynomials, and ordered perfect matchings

The aim of this paper is to give a combinatorial interpretation of Bessel polynomials and their first derivatives in terms of ordered hit polynomials.

The *Bessel polynomials*, $\theta_n(x)$, see Grosswald [5] and Carlitz [1], are given by the generating function

$$\frac{1}{\sqrt{1-2v}}\, e^{x(1-\sqrt{1-2v})} = \sum_{n=o}^{\infty} \frac{v^n}{n!}\,\theta_n(x), \tag{1}$$

and the explicit form

$$\theta_n(x) = \sum_{k=0}^{n} \theta(n,k)\, x^k = \sum_{k=0}^{n} \frac{(2n-k)!}{k!\,(n-k)!\,2^{n-k}}\, x^k, \tag{2}$$

where $\theta(n,k)$ is the coefficient of x^k in $\theta_n(x)$, the n-th Bessel polynomial.

For $n \geq 1$ consider K_{2n}, the complete graph on $2n$ vertices, with its vertices labelled by the integers $\{1,\ldots,2n\}$. Now let G be a simple graph on $m \leq 2n$ vertices labelled from $\{1,\ldots,m\}$, and with its edges colored red. Embed G into this labelled K_{2n} in the natural way, and call the graph so formed $K_{2n}|G$. Thus every edge in $K_{2n}|G$ is represented by a pair of integers (r,s), where $1 \leq r < s \leq 2n$; such an edge is called an *increasing* edge. For a red edge only we consider an interchange of its vertex labels to (s,r); a *decreasing* edge. Edges are increasing unless stated otherwise.

A *perfect matching* of $K_{2n}|G$ is a set of any n disjoint edges which cover every vertex exactly once. Suppose that such a perfect matching has exactly i edges in G, *i.e.*, exactly i increasing red edges. Then, from this perfect matching, we can form 2^i *ordered perfect matchings* by choosing any of its 2^i subsets of increasing red edges, and making each edge in the complement of this subset into a decreasing edge by interchanging its vertex labels.

For example see the graph $K_4|P_4$ in §6; here P_4 denotes the path with 4 vertices, $\{r,s\}$ denotes a red edge, and $[r,s]$ a non-red edge. The perfect matching $\{2,3\}$, $[1,4]$ has 1 increasing red edge and gives rise to the $2^1 = 2$ ordered perfect matchings: $\{2,3\}$, $[1,4]$ (with 1 increasing red edge), and $\{3,2\}$, $[1,4]$ (with 0 increasing red edges).

For $k \geq 0$ let $o(G,k)$ be the number of ordered perfect matchings in $K_{2n}|G$ with exactly k increasing red edges. Now define $\mathcal{O}(K_{2n}|G,x) = \sum_k o(G,k)x^k$, the *ordered hit polynomial* of $K_{2n}|G$.

Our main result is Theorem 3.4: if $G = P_{2n}$, the path on $2n$ vertices, then $\theta_n(x) = \mathcal{O}(K_{2n}|P_{2n},x)$, the ordered hit polynomial of $K_{2n}|P_{2n}$. Equivalently, $\theta_n(x-1) = \mathcal{H}(K_{2n}|P_{2n},x)$, the *hit polynomial* of $K_{2n}|P_{2n}$. Also, $\theta'_n(x) = \mathcal{O}(K_{2n}|P_{2n-1},x)$ and $\theta'_n(x-1) = \mathcal{H}(K_{2n}|P_{2n-1},x)$. Finally we define homogeneous Bessel polynomials, and give a combinatorial interpretation of these, and give some examples.

Note that the matchings polynomial of the path P_{2n} is known to be associated with the Chebyshev polynomials of the second kind, see Heilman and Lieb [6]. For a different combinatorial interpretation of the Bessel polynomials see Dulucq and Favreau [3], and for a combinatorial interpretation of general orthogonal polynomials see Viennot [7]. See also Bergeron, Leroux, and Labelle [2].

2 Matchings polynomials, hit polynomials, ordered hit polynomials

As in §1 let G be a simple graph on $m \leq 2n$ with its vertices labelled from $\{1,\ldots,m\}$ and with its edges colored red. For $k \geq 0$ a *k-matching* of G is a set of k edges, no two of which have a vertex in common. Let $p(G,k)$ denote the number of k-matchings of G, with $p(G,0) = 1$ for the empty matching (which contains no edges).

A perfect matching in a graph with an even number of vertices is a matching which covers every vertex exactly once. We work with K_{2n}, the complete graph on $2n$ vertices, labelled by $\{1,\ldots,2n\}$, where $n \geq 1$. In K_{2n} a perfect matching is an n-matching, and it is well-known that the number of perfect matchings, Π_n, is given by

$$\Pi_n = \frac{(2n)!}{2^n\, n!}, \quad \text{with} \quad \Pi_0 = 1.$$

Now embed G into this labelled K_{2n}, producing the graph $K_{2n}|G$.

Let $o(G,k)$ be the number of ordered perfect matchings of $K_{2n}|G$ with exactly k increasing red edges.

Lemma 2.1 *For $0 \le k \le n$, the number of ordered perfect matchings of $K_{2n}|G$ with k increasing red edges is*

$$o(G,k) = p(G,k) \cdot \Pi_{n-k}.$$

Proof. Pick a k-matching from the red G embedded in K_{2n}. The vertices not covered by this k-matching form a copy of K_{2n-2k}, which has Π_{n-k} perfect matchings. Thus each k-matching of G can be extended to Π_{n-k} ordered perfect matchings of $K_{2n}|G$ with exactly k increasing red edges, by ensuring that whenever an edge of the perfect matching is red, we make it decreasing by interchanging its vertex labels. By definition there are $p(G,k)$ k-matchings of G; hence the result by the multiplication principle. ∎

For $0 \le i \le n$ let $h(G,i)$ denote the number of perfect matchings of $K_{2n}|G$ with exactly i edges in G, *i.e.*, with exactly i increasing red edges. In order to form an ordered perfect matching with k increasing red edges from a perfect matching with i increasing red edges, we must have $k \le i$, and each such perfect matching gives rise to $\binom{i}{k}$ such ordered perfect matchings. Moreover, each ordered perfect matching with k increasing red edges (and, perhaps, some decreasing red edges) comes from only one perfect matching, that can be obtained by changing these decreasing red edges into increasing ones. We combine these observations, the inverse relation for binomial sums, and the above lemma in the following proposition.

Proposition 2.2 *We have the inverse relations*

$$o(G,k) = \sum_{i=k}^{n} \binom{i}{k} h(G,i), \; h(G,k) = \sum_{i=0}^{n} \binom{i}{k} (-1)^{k+i} p(G,i)\, \Pi_{n-i}.$$

∎

Next, define the generating polynomials

$$\mathcal{P}(G,x) = \sum_{k=0}^{n} p(G,k)\, x^k, \qquad \mathcal{H}(K_{2n}|G,x) = \sum_{i=0}^{n} h(G,i)\, x^i,$$

$$\text{and} \quad \mathcal{O}(K_{2n}|G,x) = \sum_{k=0}^{n} o(G,k)\, x^k.$$

$\mathcal{P}(G,x)$ is the *matchings polynomial* of G and $\mathcal{H}(K_{2n}|G,x)$ is the *hit polynomial* of $K_{2n}|G$. Lastly, $\mathcal{O}(K_{2n}|G,x)$ is the *ordered hit polynomial* of $K_{2n}|G$, here the coefficient of x^k is the number of ordered perfect matchings of $K_{2n}|G$ with exactly k increasing red edges.

Using Lemma 2.1 we have

$$\mathcal{O}(K_{2n}|G,x) = \sum_{k=0}^{n} p(G,k)\, \Pi_{n-k}\, x^k, \tag{3}$$

then Proposition 2.2 gives

Lemma 2.3 *The hit polynomial satisfies*

$$\mathcal{H}(K_{2n}|G,x) = \sum_{i=0}^{n} (x-1)^i p(G,i)\, \Pi_{n-i} = \mathcal{O}(K_{2n}|G,x-1).$$

■

See Godsil [4], p. 10, Lemma 4.1, for a similar derivation of the hit polynomial of $K_{n,n}$, the complete bipartite graph on $2n$ vertices.

Remark. The previous results and comments tell us that: for $k=0$, and any G with $\leq 2n$ vertices, we have

$$p(G,0) = 1, \quad o(G,0) = \Pi_n, \quad \text{and} \quad h(G,0) = \sum_{i=0}^{n} (-1)^i\, o(G,i), \tag{4}$$

and, for $k=n$, we have

$$p(G,n) = o(G,n) = h(G,n), \tag{5}$$

the number of n-matchings in G; if G has $< 2n$ vertices, then $p(G,n) = 0$.

Remark. Lemma 2.1 may be recast as an integral transform in terms of the moments of the χ^2 distribution. Recall that a χ^2 random variable with one degree of freedom, the square of a standard Gaussian, has the density

$$x^{-\frac{1}{2}}\, e^{-x/2}/\sqrt{2\pi}$$

for $0 < x < \infty$. If Y is distributed as χ^2 with one degree of freedom, then the moments are given by

$$\langle Y^n \rangle = 2^n \left(\frac{1}{2}\right)_n = \Pi_n$$

where $\langle \cdot \rangle$ denotes expected value (integration with respect to Y). Thus, Lemma 2.1 yields

Proposition 2.4 *In terms of $\mathcal{P}(G, x)$, we have*

$$\mathcal{O}(K_{2n}|G, x) = \langle Y^n \mathcal{P}(G, x/Y) \rangle,$$

where Y has a χ^2 distribution with one degree of freedom.

■

Cf. Problem [10], p. 16, of [4].

3 The graph $\mathbf{K_{2n}|P_{2n}}$, Chebyshev and Bessel polynomials

From now on our graph G will be a path. For $m \geq 2$ let P_m denote the path on m vertices with $m-1$ edges. Our focus will be on the graphs $K_{2n}|P_{2n}$ and, in the following section, $K_{2n}|P_{2n-1}$. In this section we count $o(P_{2n}, k)$ and so determine the generating polynomials $\mathcal{O}(K_{2n}|P_{2n}, x)$ and $\mathcal{H}(K_{2n}|P_{2n}, x)$ in terms of the Bessel polynomials.

Recall that $p(P_m, k)$ is the number of k-matchings in P_m; we prove the following formula for $p(P_m, k)$, cf. Godsil [4], p. 1.

Lemma 3.1 *For $m \geq 2$ we have*

$$p(P_m, k) = \binom{m-k}{k}.$$

Proof. View P_m as running from left to right. Given a k-matching, contract each of its edges to their left-hand vertex. We obtain a path on $m-k$ vertices, with k distinguished vertices. Conversely, given a path on $m-k$ vertices with k distinguished vertices, we may expand each of these to an edge, producing a k-matching of the P_m so formed.

∎

Let $\mathcal{P}(P_m, x)$ be the matchings polynomial of P_m, then

$$\mathcal{P}(P_m, x) = \sum_{k=0}^{\lfloor m/2 \rfloor} p(P_m, k)\, x^k = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} x^k. \tag{6}$$

The Chebyshev polynomials of the second kind are defined by

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k \, (2x)^{n-2k}.$$

Thus we note (see Heilman and Lieb [6])

Corollary 3.2 *The matchings polynomial of the path P_{2n} is given by,*

$$\mathcal{P}(P_{2n}, x) = (-1)^n x^n \, U_{2n}\left(\frac{i}{2\sqrt{x}}\right),$$

in terms of the Chebyshev polynomials of the second kind (here $i = \sqrt{-1}$).

∎

Now for the Bessel polynomials,

Lemma 3.3 *The coefficient of x^k in $\theta_n(x)$ is given by*

$$\theta(n, k) = o(P_{2n}, k).$$

Proof. From (2) and Lemma 2.1, we have

$$\theta(n, k) = \frac{(2n-k)!}{k!(2n-2k)!} \cdot \frac{(2n-2k)!}{2^{n-k}(n-k)!} = p(P_{2n}, k) \cdot \Pi_{n-k} = o(P_{2n}, k).$$

∎

This gives our main result:

Theorem 3.4 *For $n \geq 1$ the n-th Bessel polynomial equals the ordered hit polynomial of $K_{2n}|P_{2n}$, i.e.,*

$$\theta_n(x) = \mathcal{O}(K_{2n}|P_{2n}, x).$$

Proof. From Lemma 3.3 we see that $\theta(n,k) = o(P_{2n}, k)$, the coefficient of x^k in $\mathcal{O}(K_{2n}|P_{2n}, x)$. This is true for all k with $0 \leq k \leq n$, hence the result. ∎

Furthermore, Lemma 2.3 implies

Theorem 3.5 *For $n \geq 1$ the n-th Bessel polynomial is related to the hit polynomial of $K_{2n}|P_{2n}$ by*

$$\theta_n(x-1) = \mathcal{H}(K_{2n}|P_{2n}, x).$$

∎

Remark. Another connection with the χ^2 distribution is evident here, as the moment generating function for the χ^2 distribution is $\frac{1}{\sqrt{1-2v}}$. I.e., with $x = 0$ in equation (1), we recover the values $\theta_n(0) = \Pi_n = o(P_{2n}, 0)$, the number of ordered perfect matchings in $K_{2n}|P_{2n}$ with no increasing red edges (put $G = P_{2n}$ in (4)).

4 The graph $\mathbf{K_{2n}|P_{2n-1}}$ and the first derivatives of the Bessel polynomials

Now we consider $K_{2n}|P_{2n-1}$, and so determine the generating polynomials $\mathcal{O}(K_{2n}|P_{2n-1}, x)$ and $\mathcal{H}(K_{2n}|P_{2n-1}, x)$ in terms of the first derivative of the n-th Bessel polynomial.

Let $\theta'(n,k)$ be the coefficient of x^k in $\theta'_n(x)$.

Lemma 4.1 *The coefficient of x^k in $\theta'_n(x)$ is given by*

$$\theta'(n,k) = o(P_{2n-1}, k).$$

Proof. Using derivatives, for $0 \le k < n$, we have

$$\begin{aligned}
\theta'(n,k) &= (k+1)\,\theta(n,k+1) \\
&= \frac{(2n-1-k)!}{k!\,(n-k-1)!\,2^{n-k-1}} \cdot \frac{2}{2} \cdot \frac{n-k}{n-k} \cdot \frac{(2n-2k-1)!}{(2n-1-2k)!} \\
&= \frac{(2n-1-k)!}{k!\,(2n-1-2k)!} \cdot \frac{(2n-2k)!}{(n-k)!\,2^{n-k}} \\
&= p(P_{2n-1},k) \cdot \Pi_{n-k} = o(P_{2n-1},k).
\end{aligned}$$

■

Now $o(P_{2n-1},k)$ is the coefficient of x^k in $\mathcal{O}(K_{2n}|P_{2n-1},x)$, so the above lemma yields (cf. Theorem 3.4)

Theorem 4.2 *For $n \ge 1$ the derivative of the n-th Bessel polynomial equals the ordered hit polynomial of $K_{2n}|P_{2n-1}$, i.e.,*

$$\theta_n'(x) = \mathcal{O}(K_{2n}|P_{2n-1},x) = \mathcal{O}'(K_{2n}|P_{2n},x).$$

■

For the hit polynomial $\mathcal{H}(K_{2n}|P_{2n-1},x)$ we have

Theorem 4.3 *For $n \ge 1$ the derivative of the n-th Bessel polynomial is related to the hit polynomial of $K_{2n}|P_{2n-1}$ by*

$$\theta_n'(x-1) = \mathcal{H}(K_{2n}|P_{2n-1},x) = \mathcal{H}'(K_{2n}|P_{2n},x).$$

■

Remark. Some comments on the first, and the last two, term(s) of $\theta_n(x)$.

We see that the coefficient of x^n in $\theta_n(x)$, $o(P_{2n},n)$, is 1. From equation (5) we have $o(P_{2n},n) = p(P_{2n},n)$, the number of n-matchings of P_{2n}, which is 1. (Similarly, the coefficient of x^n in $\theta_n'(x)$ is 0 and $o(P_{2n-1},n) = p(P_{2n-1},n)$, the number of n-matchings of P_{2n-1}, is 0.)

The coefficients of the last two terms of $\theta_n(x)$ are both equal to Π_n. We can explain this: from equation (4) we have $o(P_{2n},0) = o(P_{2n-1},0) = \Pi_n$. Also, as $\mathcal{O}(K_{2n}|P_{2n-1},x) = \mathcal{O}'(K_{2n}|P_{2n},x)$, then $o(P_{2n-1},0) = o(P_{2n},1)$; and so $o(P_{2n},1) = o(P_{2n},0) = \Pi_n$ as required.

5 Homogeneous Bessel polynomials and completely ordered hit polynomials

For $n \geq 1$ let $h_n(x) = \theta_n(x-1) = \mathcal{H}(K_{2n}|P_{2n}, x)$ be the hit polynomial of $K_{2n}|P_{2n}$. Define a homogeneous version of $h_n(x)$, call it $H_n(X, Z)$,

$$H_n(X, Z) = \sum_{i=0}^{n} h(P_{2n}, i) X^i Z^{n-i} = Z^n h_n\left(\frac{X}{Z}\right),$$

with $H_0(X, Z) = 1$.

It is straightforward to show that, for $n \geq 0$, $H_n(X, Z)$ is given in terms of the n-th Bessel polynomial by

$$H_n(X, Z) = Z^n \theta_n\left(\frac{X-Z}{Z}\right). \tag{7}$$

The coefficient of $X^i Z^{n-i}$ in $H_n(X, Z)$ is the number of perfect matchings in $K_{2n}|P_{2n}$ with exactly i red edges and $n-i$ non-red edges.

A *completely ordered perfect matching* of $K_{2n}|P_{2n}$ is obtained from a perfect matching of $K_{2n}|P_{2n}$ by choosing a subset of its red edges, and also a subset of its non-red edges, to be increasing. Each perfect matching gives rise to 2^n completely ordered perfect matchings.

Considering the red edges, a completely ordered perfect matching with i red edges can have a increasing red edges and $b = i-a$ decreasing red edges, for some a, with $0 \leq a \leq i$. The a increasing red edges can be chosen from the i red edges in $\binom{i}{a}$ ways, which is the coefficient of $x^a y^b$ in $(x+y)^i$. Hence, the number of completely ordered perfect matchings with a increasing red edges, b decreasing red edges, and $n-i$ non-red edges is the coefficient of $x^a y^b Z^{n-i}$ in $H_n(x+y, Z)$. Similarly for the non-red edges.

Let $\mathcal{S}(K_{2n}|P_{2n}; x, y, z, w)$ be the generating function for completely ordered perfect matchings of $K_{2n}|P_{2n}$, the *completely ordered hit polynomial* of $K_{2n}|P_{2n}$. So the coefficient of $x^a y^b z^c w^d$ in $\mathcal{S}(K_{2n}|P_{2n}; x, y, z, w)$ is the number of completely ordered perfect matchings of $K_{2n}|P_{2n}$ with a increasing red edges, b decreasing red edges, c increasing non-red edges, and d decreasing non-red edges.

Then, by the previous comments and equation (7),

Theorem 5.1 *For $n \geq 1$ the completely ordered hit polynomial of $K_{2n}|P_{2n}$ is given by*

$$\begin{aligned}\mathcal{S}(K_{2n}|P_{2n};x,y,z,w) &= H_n(x+y,z+w) \\ &= (z+w)^n\theta_n\left(\frac{(x+y)-(z+w)}{z+w}\right).\end{aligned}$$

■

Now, as before, consider the graph $K_{2n}|P_{2n-1}$, where the derivative of the n-th Bessel polynomial comes into play.

Let $\mathcal{S}(K_{2n}|P_{2n-1};x,y,z,w)$ be the generating function for completely ordered perfect matchings of $K_{2n}|P_{2n-1}$. Then, by an argument similar to the above, we have our final theorem.

Theorem 5.2 *For $n \geq 1$ the completely ordered hit polynomial of $K_{2n}|P_{2n-1}$ is given by*

$$\mathcal{S}(K_{2n}|P_{2n-1};x,y,z,w) = (z+w)^n\theta_n'\left(\frac{(x+y)-(z+w)}{z+w}\right).$$

■

In later work we investigate a combinatorial interpretation of the higher derivatives of the Bessel polynomials, this requires different graphs than K_{2n} and different techniques than the counting of Lemma 2.1.

6 Examples

The first few Bessel polynomials are:
$\theta_0 = 1,\quad \theta_1 = x+1,\quad \theta_2 = x^2+3x+3,$
$\theta_3 = x^3+6x^2+15x+15,\ \theta_4 = x^4+10x^3+45x^2+105x+105,$
and the first few Chebyshev polynomials of the second kind are:
$U_0(x) = 1,\quad U_1(x) = 2x,\quad U_2(x) = 4x^2-1,$
$U_3(x) = 8x^3-4x,\quad U_4(x) = 16x^4-12x^2+1.$

Consider the path P_4

From Corollary 3.2 the matchings polynomial of P_4 is given by

$$\mathcal{P}(P_4, x) = (-1)^2 x^2 \, U_4\left(\frac{i}{2\sqrt{x}}\right) = 1 + 3x + x^2.$$

The 1 corresponds to the empty matching, the $3(x^1)$ to the 3 1-matchings, *i.e.*, the 3 edges, and the $1(x^2)$ to the 1 2-matching consisting of the first and last edges. From (6), we also have

$$\mathcal{P}(P_4, x) = \sum_{k=0}^{2} \binom{4-k}{k} x^k = 1 + 3x + x^2,$$

so, from (3), we have

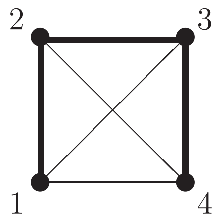
$$\mathcal{O}(K_4|P_4, x) = \Pi_2 + 3\Pi_1 x + \Pi_0 x^2 = 3 + 3x + x^2 = \theta_2(x).$$

Similarly, the matchings polynomial of P_3 is $\mathcal{P}(P_3, x) = 1 + 2x$, which gives

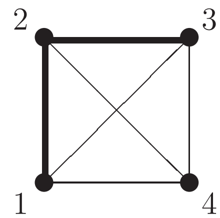
$$\mathcal{O}(K_4|P_3, x) = \Pi_2 + 2\Pi_1 x = 3 + 2x = \theta_2'(x),$$

see below for examples.

In the following examples $\{r, s\}$ denotes a red edge (thick lines in the figures), and $[r, s]$ a non-red edge (indicated by thin lines).



$K_4|P_4$



$K_4|P_3$

$$\begin{aligned}\theta_2(x-1) &= \mathcal{H}(K_4|P_4, x)\\ &= x^2 + x + 1.\end{aligned}$$

$\{1,2\}, \{3,4\}$	x^2
$\{2,3\}, [1,4]$	x
$[1,3], [2,4]$	1

perfect matchings

$$\begin{aligned}\theta_2'(x-1) &= \mathcal{H}(K_4|P_3, x)\\ &= 2x + 1.\end{aligned}$$

$\{1,2\}, [3,4]$	x
$\{2,3\}, [1,4]$	x
$[1,3], [2,4]$	1

perfect matchings

$$\begin{aligned}\theta_2(x) &= \mathcal{O}(K_4|P_4, x)\\ &= x^2 + 3x + 3.\end{aligned}$$

$\{1,2\}, \{3,4\}$	x^2
$\{1,2\}, \{4,3\}$	x
$\{2,1\}, \{3,4\}$	x
$\{2,1\}, \{4,3\}$	1
$\{2,3\}, [1,4]$	x
$\{3,2\}, [1,4]$	1
$[1,3], [2,4]$	1

ordered perfect matchings

$$\begin{aligned}\theta_2'(x) &= \mathcal{O}(K_4|P_3, x)\\ &= 2x + 3.\end{aligned}$$

$\{1,2\}, [3,4]$	x
$\{2,1\}, [3,4]$	1
$\{2,3\}, [1,4]$	x
$\{3,2\}, [1,4]$	1
$[1,3], [2,4]$	1

ordered perfect matchings

$\mathcal{S}(K_4\|P_4; x, y, z, w)$ $= x^2 + 2xy + xz$ $+xw + y^2 + yz + yw$ $+z^2 + 2zw + w^2.$		$\mathcal{S}(K_4\|P_3; x, y, z, w)$ $= 2xz + 2xw + 2yz$ $+2yw + z^2 + 2zw + w^2.$	
$\{1,2\}, \{3,4\}$	x^2	$\{1,2\}, [3,4]$	xz
$\{1,2\}, \{4,3\}$	xy	$\{1,2\}, [4,3]$	xw
$\{2,1\}, \{3,4\}$	xy	$\{2,1\}, [3,4]$	yz
$\{2,1\}, \{4,3\}$	y^2	$\{2,1\}, [4,3]$	yw
$\{2,3\}, [1,4]$	xz	$\{2,3\}, [1,4]$	xz
$\{2,3\}, [4,1]$	xw	$\{2,3\}, [4,1]$	xw
$\{3,2\}, [1,4]$	yz	$\{3,2\}, [1,4]$	yz
$\{3,2\}, [4,1]$	yw	$\{3,2\}, [4,1]$	yw
$[1,3], [2,4]$	z^2	$[1,3], [2,4]$	z^2
$[1,3], [4,2]$	zw	$[1,3], [4,2]$	zw
$[3,1], [2,4]$	zw	$[3,1], [2,4]$	zw
$[3,1], [4,2]$	w^2	$[3,1], [4,2]$	w^2
completely ordered perfect matchings		completely ordered perfect matchings	

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