Constructing and Classifying Neighborhood Anti-Sperner Graphs

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October 1, 2007

Abstract

For a simple graph G let $N_G(u)$ be the (open) neighborhood of vertex $u \in V(G)$. Then G is neighborhood anti-Sperner (NAS) if for every u there is a $v \in V(G)\backslash\{u\}$ with $N_G(u) \subseteq N_G(v)$. And a graph H is neighborhood distinct (ND) if every neighborhood is distinct, *i.e.*, if $N_H(u) \neq N_H(v)$ when $u \neq v$, for all u and $v \in V(H)$.

In Porter and Yucas [3] a characterization of *regular* NAS graphs was given: 'each regular NAS graph can be obtained from a host graph by replacing vertices by null graphs of appropriate sizes, and then joining these null graphs in a prescribed manner'. We extend this characterization to *all* NAS graphs, and give algorithms to construct all NAS graphs from host ND graphs. Then we find and classify all connected r-regular NAS graphs for $r = 0, 1, \ldots, 6$.

Keywords: Graph, Neighborhood, Distinct, Sperner, anti-Sperner, Classify.

1 Introduction and Main Results

We first give some definitions from Porter [2], Porter and Yucas [3], and Sumner [5]. Standard definitions of graph theory are from West [6].

Let $\mathcal{F} = \{S_1, S_2, \ldots\}$ be a family of sets. Then $\mathcal{F}$ is *Sperner* if no member of $\mathcal{F}$ is a subset of another member; and $\mathcal{F}$ is *anti-Sperner* if *every* member of $\mathcal{F}$ is a subset of another member.

Let G be a simple graph, with a finite number of vertices. For each $u \in V(G)$ let $N_G(u)$ denote the open neighborhood of u, *i.e.*, the set of vertices that u is adjacent to. Here we use 'neighborhood' for 'open neighborhood'.

Let $\mathcal{F}(G) = \{N_G(u) \,|\, u \in V(G)\}$ be the family of neighborhoods of G. Then G is *neighborhood anti-Sperner* (NAS) if $\mathcal{F}(G)$ is anti-Sperner. Hence, in a NAS graph G, for every $u \in V(G)$ there is a $u^p \in V(G)\backslash\{u\}$ such that $N_G(u) \subseteq N_G(u^p)$. We say that u^p is a *parent* of u and note that u is not adjacent to u^p. Neighborhood anti-Sperner graphs were introduced by Porter in [2] where they were called ANS graphs, and studied further in Porter and Yucas [3].

A graph H is *neighborhood distinct* (ND) if every neighborhood is distinct, *i.e.*, if $N_H(u) \neq N_H(v)$ when $u \neq v$, for all $u, v \in V(H)$.

Let H be an arbitrary graph with t vertices and let $T_1, T_2, \ldots, T_t$ be t arbitrary graphs. Using the notation of [3], the graph $G = I(T_1, T_2, \ldots, T_t : H)$ is the graph obtained from H by replacing vertex i with a copy of T_i for each $i = 1, 2, \ldots, t$, and then if $ij \in E(H)$ joining T_i to T_j. In Equation (1) Section 3 of [3] a characterization of *regular* NAS graphs is given which we reproduce here with a mix of their and our notations, where N_ℓ is the null graph with $\ell \geq 1$ vertices and no edges:

> *Let G be a regular NAS graph with reduced graph $G_\equiv$ on t vertices with labels $\ell_1, \ell_2, \ldots, \ell_t$. Then $G = I(N_{\ell_1}, N_{\ell_2}, \ldots, N_{\ell_t} : G_\equiv)$.*

A direct result of our Theorem 3.2 is an extension of this result from regular NAS graphs to arbitrary NAS graphs. We comment on this after Theorem 3.2, and write this up formally below and in our Extensions and Conclusions (Section 5), as Theorem 5.1.

Theorem 5.1 *Let G be a NAS graph with reduced graph $G_\equiv$ on t vertices with labels $\ell_1, \ell_2, \ldots, \ell_t$. Then $G = I(N_{\ell_1}, N_{\ell_2}, \ldots, N_{\ell_t} : G_\equiv)$.*

Using this extension we give an Algorithm to construct all NAS graphs on a fixed number of $n \geq 2$ vertices from labelled ND graphs on $\leq n - 1$

vertices, and then an Algorithm to construct all connected NAS graphs on a fixed number of $n \geq 4$ vertices from connected labelled ND graphs on $\leq n-2$ vertices.

In [3] all connected r-regular NAS graphs were found and classified for $r = 2, 3,$ and 4. In Section 4 we simplify this classification and extend it to $r = 5$ and 6. We also extend Theorem 2.11 of [3], see Theorem 5.2 in Section 5. Section 2 next contains preparatory material.

Some results of [3] concern infinite NAS graphs, however in this paper we only consider *finite* NAS graphs.

2 Neighborhood distinct graphs, labelled graphs, miscellaneous,

This Section contains preparatory and miscellaneous material required in later Sections.

The join $X \vee Y$ of two graphs X and Y with disjoint vertex sets is the graph with vertex set $V(X) \cup V(Y)$ and edge set $E(X) \cup E(Y) \cup \{xy \,|\, x \in V(X)$ and $y \in V(Y)\}$, *i.e.*, every vertex in $V(X)$ is joined to every vertex in $V(Y)$.

Recall that a graph H is neighborhood distinct (ND) if every neighborhood is distinct, *i.e.*, if $N_H(u) \neq N_H(v)$ when $u \neq v$, for all $u, v \in V(H)$. Sumner [5] called such graphs *point determining* and proved the following Theorem that we write using our notation, (see Theorem 2 of [5]):

> *Let H be a ND graph with ≥ 2 vertices. Then there is a vertex $w \in V(H)$ for which $H - w$ is also ND.*

The ND graphs with ≤ 3 vertices are: N_1, K_2, $N_1 \cup K_2$, and K_3.

Sumner's result gives us:

Algorithm ND Graphs Four step algorithm to construct all ND graphs H on a fixed number of $t \geq 2$ vertices from all ND graphs on $t-1$ vertices.

(1) `List all non-isomorphic ND graphs` H_{t-1} `on` $t-1$ `vertices.`

(2) `For each` H_{t-1} `list all subsets` $S \subseteq V(H_{t-1})$ `for which` $S \neq N_{H_{t-1}}(u)$ `for all` $u \in V(H_{t-1})$`,` *i.e.*`,` S `is distinct from all neighborhoods of` H_{t-1}`. We allow` $S = \emptyset$`.`

(3) `Let` $w \notin V(H_{t-1})$ `be a new vertex. For each such` H_{t-1} `and` S `let` H `be the graph obtained by joining` w `to` S`:`

$$V(H) = V(H_{t-1}) \cup \{w\} \qquad \texttt{and} \qquad E(H) = E(H_{t-1}) \cup \{ws \,|\, s \in S\}.$$

(4) `Remove non ND graphs and then isomorphic copies of the remaining ND graphs from the list in (3).`

We now have a complete list of ND graphs H on t vertices.

Example 1 We find all ND graphs with 4 vertices from the 2 ND graphs $N_1 \cup K_2$ and K_3 with 3 vertices. For each such graph there are $2^3 - 3 = 5$ subsets S, yielding 10 graphs at step (3). Removing non ND graphs and isomorphic copies of the remaining ND graphs leaves the 5 non-isomorphic ND graphs with 4 vertices:

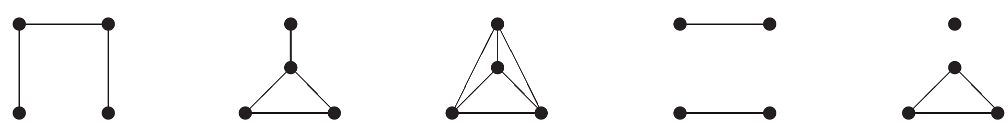


Fig. 1. All ND graphs with 4 vertices

We can then use these ND graphs on 4 vertices to construct all ND graphs on 5 vertices, ... , and so on. Hence, for any $t \geq 1$, we can construct all ND graphs on $\leq t$ vertices.

We will need *labelled* graphs H in which every vertex $u \in V(H)$ has been labelled with a positive integer $\ell(u) \geq 1$. We also need the concept of label-isomorphism:

Let H and H' be two arbitrary labelled graphs. Then H and H' are *label-isomorphic* if there is a graph isomorphism between H and H' that preserves vertex labels. So if in a label-isomorphism we have $u \in V(H) \leftrightarrow u' \in V(H')$, then $\ell(u) = \ell(u')$.

If $N_G(u) = N_G(v)$ for two different vertices u and $v \in V(G)$ then we call u and v *twins*. We note that twins are non-adjacent. We denote a twin of u by u^*. If u doesn't have a twin then it is *twinless*. If H is ND then every vertex of H is twinless. Graphs that we call ND have also been considered in Kotlov and Lovász [1] under the name 'twin-free'. In a NAS graph a vertex of maximum degree must have a twin. The smallest NAS graph is N_2 with 2 vertices, and the smallest connected NAS graph is $K_{2,2}$ with 4 vertices.

3 Constructing NAS Graphs from labelled ND Graphs

In this Section we show how to construct all NAS graphs G on $n \geq 2$ vertices from labelled ND graphs H on $\leq n-1$ vertices, and all connected NAS graphs G on $n \geq 4$ vertices from connected labelled ND graphs H on $\leq n-2$ vertices.

Let G be an arbitrary graph. Consider the following equivalence relation $\equiv$ on $V(G)$: $u \equiv u'$ if and only if $N_G(u) = N_G(u')$. The equivalence class containing u is $U = \{u' \in V(G) \mid N_G(u) = N_G(u')\} \neq \emptyset$. Here every vertex u' is a twin of u, we normally write a twin as u^* provided that it is distinct from u. We let t denote the number of equivalence classes of $V(G)$ or of G under $\equiv$; and denote the classes themselves by $U_1, U_2, \ldots, U_t$, where $|U_i| = \ell_i$ for each $i = 1, 2, \ldots, t$.

Theorem 3.1 *Let G be an arbitrary graph with equivalence relation $\equiv$. Let U and V be two distinct equivalence classes with $u \in U$ and $v \in V$ arbitrary. Then*

(i) the induced subgraph $G[U] = N_{|U|}$,

(ii) $uv \in E(G)$ if and only if $G[U \cup V] = N_{|U|} \vee N_{|V|}$,

(iii) $uv \notin E(G)$ if and only if $G[U \cup V] = N_{|U|} \cup N_{|V|}$.

Proof. (i) If $|U| = 1$ then clearly $G[U] = N_{|U|}$. So assume that $|U| \geq 2$ and let u and u^* be two arbitrary distinct vertices in U. If $uu^* \in E(G)$ then $u \in N_G(u^*) = N_G(u)$, a contradiction since G is simple. Hence $uu^* \notin E(G)$, and so $G[U] = N_{|U|}$.
(ii) Now $u \in U$ and $v \in V$ are arbitrary, let $u' \in U$ and $v' \in V$ also be arbitrary, (so $u = u'$ and/or $v = v'$ is allowed). Now $uv \in E(G)$, so $v \in N_G(u) = N_G(u')$, so $u' \in N_G(v) = N_G(v')$, and then $u'v' \in E(G)$. Hence $G[U \cup V] = G[U] \vee G[V] = N_{|U|} \vee N_{|V|}$. The converse is clear.
(iii) Straightforward. ∎

So, in any graph G, and for any two distinct equivalence classes U and V either $G[U \cup V] = N_{|U|} \vee N_{|V|}$ or $G[U \cup V] = N_{|U|} \cup N_{|V|}$. This suggests the following two constructions:

Construction $\mathbf{G}_\equiv$ Let G be an arbitrary graph, with equivalence relation $\equiv$ and equivalence classes $U_1, U_2, \ldots, U_t$, where $|U_i| = \ell_i$ for each $i = 1, 2, \ldots, t$. Construct a labelled graph $G_\equiv$ with t vertices and edges as follows:

$$V(G_\equiv) = \{U_1, U_2, \ldots, U_t\} \quad \text{and} \quad E(G_\equiv) = \{U_iU_j \mid G[U_i \cup U_j] = N_{|U_i|} \vee N_{|U_j|}\},$$

where vertex U_i has been labelled with ℓ_i for each i. Note that $|V(G)| = \sum_{i=1}^{t} \ell_i$.

We call $G_\equiv$ the *reduced* graph of G. An unlabelled version of $G_\equiv$ is the host graph from Section 3 of [3], although here G was assumed to be a regular NAS graph.

Construction $\mathbf{H}^\uparrow$ Let H be an arbitrary labelled graph, so every $u \in V(H)$ has been labelled with a positive integer $\ell(u) \geq 1$. Construct a new graph $H^\uparrow$ from H by expanding each vertex u as follows: replace u with the $\ell(u)$ vertices from $exp(u) = \{x_1, x_2, \ldots, x_{\ell(u)}\}$, the *expansion set* of u, where $H^\uparrow[exp(u)] = N_{\ell(u)}$. If $uv \in E(H)$ then let $H^\uparrow[exp(u) \cup exp(v)] = N_{\ell(u)} \vee N_{\ell(v)}$, and if $uv \notin E(H)$ then let $H^\uparrow[exp(u) \cup exp(v)] = N_{\ell(u)} \cup N_{\ell(v)}$. (The graph $H^\uparrow$ is sometimes referred to as the *blow-up* of H.)

From the above two constructions we have:

Theorem 3.2 *Let G be an arbitrary graph. Then $G = (G_\equiv)^\uparrow$.* ∎

We obtain our first result by applying Theorem 3.2 to arbitrary NAS graphs. This is an extension of the result of Equation (1) Section 3 of [3] from regular NAS graphs to arbitrary NAS graphs. We write this up formally in our Extensions and Conclusions (Section 5), see Theorem 5.1.

Now we proceed with the main topic of this Section.

Given an arbitrary graph G, as we reduce to $G_{\equiv}$ we identity vertices with the same neighborhood, so $G_{\equiv}$ should be ND:

Theorem 3.3 *Let G be an arbitrary graph. Then $G_{\equiv}$ is ND.*

Proof. Let U and V be two distinct vertices in $V(G_{\equiv})$. Suppose that $G_{\equiv}$ is not ND and $N_{G_{\equiv}}(U) = N_{G_{\equiv}}(V) = \{U_1, U_2, \ldots, U_d\}$ for some $d \geq 1$, or $N_{G_{\equiv}}(U) = N_{G_{\equiv}}(V) = \emptyset$.

In the first case let $u \in U$ then $N_G(u) = \bigcup_{k=1}^{d} U_k$. Similarly, if $v \in V$ then $N_G(v) = \bigcup_{k=1}^{d} U_k$. Hence $N_G(u) = N_G(v)$ so $u \equiv v$, a contradiction. The proof is similar when $N_{G_{\equiv}}(U) = N_{G_{\equiv}}(V) = \emptyset$. ■

The following two technical Lemmas are required before our main results:

Lemma 3.4 *Let H be a labelled ND graph with $t \geq 1$ vertices. Then $H^{\uparrow}$ has t equivalence classes under $\equiv$.*

Proof. Let $H^{\uparrow}$ have s equivalence classes under $\equiv$, we show that $s = t$.

Let each vertex $u \in V(H)$ be labelled with $\ell(u) \geq 1$. The Lemma is clearly true if $t = 1$. So assume that $t \geq 2$ and let u and v be distinct vertices in $V(H)$. In the construction of $H^{\uparrow}$ from H we replace u by the $\ell(u)$ vertices from $exp(u) = \{x_1, x_2, \ldots, x_{\ell(u)}\}$, and v by the $\ell(v)$ vertices from $exp(v) = \{y_1, y_2, \ldots, y_{\ell(v)}\}$. Let $x_i \in exp(u)$ and $y_j \in exp(v)$ be arbitrary. Now, since H is ND, we have $N_H(u) \neq N_H(v)$. Without loss of generality let $w \in N_H(u) \backslash N_H(v)$ and let $exp(w) = \{z_1, z_2, \ldots, z_{\ell(w)}\}$, ($w = v$ is allowed). Then, in $H^{\uparrow}$, we have $x_i \in N_{H^{\uparrow}}(z_1)$ but $y_j \notin N_{H^{\uparrow}}(z_1)$. So $N_{H^{\uparrow}}(x_i) \neq N_{H^{\uparrow}}(y_j)$, and so $x_i \not\equiv y_j$ in $H^{\uparrow}$. So x_i and y_j are in distinct equivalence classes of $H^{\uparrow}$. Now let $V(H) = \{u_1, u_2, \ldots, u_t\}$. We can apply the above argument to every distinct pair u_a and $u_b \in V(H)$, showing that $exp(u_a)$ and $exp(u_b)$ are contained in distinct equivalence classes of $H^{\uparrow}$. (A slight modification of this argument is required if $w = v$.) Hence $t \leq s$.

Suppose $s > t$. Let $\{e_1, e_2, \ldots, e_s\}$ be representatives of the s equivalence classes under $\equiv$ in $H^{\uparrow}$, one from each class. Then, by the pigeon hole principle, there must be some vertex $u \in V(H)$ whose expansion set $exp(u)$ contains two of $\{e_1, e_2, \ldots, e_s\}$. Suppose that e_a and $e_b \in exp(u)$ for some

$a \neq b$, then $N_{H^\uparrow}(e_a) = N_{H^\uparrow}(e_b)$, *i.e.*, $e_a \equiv e_b$ in $H^\uparrow$, a contradiction. Hence $s \leq t$. And so $s = t$. ■

In an arbitrary graph G we say that vertex $u \in V(G)$ is *parentless* if u does not have a parent. And if u does have a parent u^p with $N_G(u) \subset N_G(u^p)$ then we call u^p a *proper parent* of u.

In Lemma 3.5, as usual, we denote the equivalence class under $\equiv$ containing u by U, and the equivalence class containing u^p by U^p.

Lemma 3.5

(i) In an arbitrary graph G let u^p be a proper parent of u. Then, in $G_\equiv$, U^p is a proper parent of U.

(ii) For a NAS graph G let $W \in V(G_\equiv)$ be parentless. Then $\ell(W) \geq 2$.

Proof. (i) In G since u^p is a proper parent of u then $N_G(u^p) \neq N_G(u)$, and so $U^p \neq U$, *i.e.*, in $G_\equiv$ the vertices U^p and U are distinct. Furthermore if $U^pU \in E(G_\equiv)$ then $u^pu \in E(G)$, a contradiction, hence $U^pU \notin E(G_\equiv)$, *i.e.*, $U^p \notin N_{G_\equiv}(U)$.

We first show that U^p is a parent of U. If not, then there exists a vertex $V \neq U^p$ (since $U^p \notin N_{G_\equiv}(U)$) with $V \in N_{G_\equiv}(U)$ but $V \notin N_{G_\equiv}(U^p)$. Now let $v \in V(G)$ lie in equivalence class V. Then $v \in N_G(u)$ but $v \notin N_G(u^p)$, a contradiction since u^p is a (proper) parent of u. So, in $G_\equiv$, U^p is a parent of U. Now $G_\equiv$ is ND so U^p cannot be a twin of U, but it is a parent of U, so it is a proper parent of U.
(ii) Let $W \in V(G_\equiv)$ be parentless, then W has no proper parents in $G_\equiv$. Let $w \in V(G)$ lie in equivalence class W, so, by (i), w has no proper parents in G. But G is NAS so w must have a parent that must be a twin w^*, so $|W| \geq 2$, *i.e.*, $\ell(W) \geq 2$. ■

The following result deals with both connected and disconnected NAS graphs.

Theorem 3.6 *Let G be an arbitrary graph. Then G is a NAS graph with t equivalence classes under $\equiv$ if and only if $G_\equiv$ is a labelled t vertex ND graph in which all parentless vertices have label ≥ 2.*

Proof. First let G be a NAS graph with t equivalence classes under $\equiv$ given by $U_1, U_2, \ldots, U_t$, where $|U_i| = \ell_i$ for each $i = 1, 2, \ldots, t$. Then the construction of $G_\equiv$ from G and Theorem 3.3 shows that $G_\equiv$ is a labelled t vertex ND graph. From Lemma 3.5(ii) all parentless vertices in $G_\equiv$ have label ≥ 2.

Conversely suppose that $G_\equiv$ is a labelled t vertex ND graph in which all parentless vertices have label ≥ 2. From Theorem 3.2 we have $G = (G_\equiv)^\uparrow$. Now any vertex $u \in V(G)$ is a $u_j \in exp(U) = \{u_1, u_2, \ldots, u_{\ell(U)}\}$ for some $U \in V(G_\equiv)$, and the neighborhoods $N_G(u_j)$ for $j = 1, 2, \ldots, \ell(U)$ are all equal. Either $\ell(U) = 1$ or $\ell(U) \geq 2$. If $\ell(U) = 1$ then U is not parentless and so U has a parent U^p, and then u_j has a parent in $exp(U^p)$. If $\ell(U) \geq 2$, then each u_j has a twin, which is a parent. Hence, in either case, $u = u_j$ has a parent, and so G is NAS. Furthermore, since $G_\equiv$ is a t vertex ND graph then, from Lemma 3.4, the graph $G = (G_\equiv)^\uparrow$ has t equivalence classes under $\equiv$. ∎

We need another definition: Let $n \geq 2$ be a positive integer. A *partition* of n is a set $\mathcal{P} = \{\ell_1, \ell_2, \ldots, \ell_t\}$ of $t \geq 1$ integers that satisfy $1 \leq \ell_1 \leq \ell_2 \cdots \leq \ell_t$ and $\sum_{i=1}^t \ell_i = n$. Partition $\mathcal{P}$ has t parts.

We now present an algorithm to construct NAS graphs G from labelled ND graphs H. It relies on Theorem 3.6: we denote $G_\equiv$ by H, and consider all possible labelled ND graphs H, and then construct all possible NAS graphs G by using $G = H^\uparrow$.

Let the labels on the t vertices of $H = G_\equiv$ be $\{\ell_1, \ell_2, \ldots, \ell_t\}$, where each $\ell_i \geq 1$. If G is NAS with $n \geq 2$ vertices then a vertex $u \in V(G)$ of maximum degree must have a twin, so $|U| \geq 2$. So some $\ell_i \geq 2$, and since $n = \sum_{i=1}^t \ell_i$, then $t \leq n - 1$.

Algorithm NAS Graphs Four step algorithm to construct all NAS graphs G on a fixed number of $n \geq 2$ vertices from all labelled ND graphs H on $t \leq n-1$ vertices.

`For each fixed` $t = 1, 2, \ldots, n-1$`:`

(1) `By repeated use of Algorithm ND Graphs list all non-isomorphic ND graphs` H_t `on` t `vertices.`

(2) `List all partitions` $\mathcal{P}_t$ `of` n `with` t `parts.`

(3) `For each graph` H_t `and partition` $\mathcal{P}_t = \{\ell_1, \ell_2, \ldots, \ell_t\}$ `label its` t `vertices with` $\{\ell_1, \ell_2, \ldots, \ell_t\}$ `in all possible non-label-isomorphic ways, ensuring that all parentless vertices have label` ≥ 2.

(4) `For each labelled graph` H_t `construct` $G = {H_t}^{\uparrow}$.

Because of Theorem 3.6 we have a complete list of NAS graphs G with n vertices, with no repeated G.

We are primarily interested in connected NAS graphs and so we now modify Algorithm NAS Graphs to generate connected NAS graphs; we require three more results.

Lemma 3.7 *Let G be an arbitrary graph. Then G is connected with ≥ 2 vertices if and only if $G_{\equiv}$ is connected with ≥ 2 vertices.*

Proof. Let G be connected with ≥ 2 vertices, and let $uv \in E(G)$. Then, since $u \notin N_G(u)$ and $u \in N_G(v)$, so $N_G(u) \neq N_G(v)$, and so $G_{\equiv}$ has ≥ 2 vertices. To see that $G_{\equiv}$ is connected, let U and V be two different vertices of $G_{\equiv}$, and let $u \in U$ and $v \in V$ in G. Then, since G is connected, there is a path $u = w_1 w_2 \cdots w_d = v$ between u and v in G, but then $U = W_1 W_2 \cdots W_d = V$ is a walk between U and V in $G_{\equiv}$, and so $G_{\equiv}$ is connected. The converse is proved similarly. ∎

Now any NAS graph G has ≥ 2 vertices, so using Lemma 3.7 we have the following 'connected' version of Theorem 3.6:

Theorem 3.8 *Let G be an arbitrary graph. Then G is a connected NAS graph with t equivalence classes under $\equiv$ if and only if $G_{\equiv}$ is a connected labelled t vertex ND graph in which all parentless vertices have label ≥ 2.* ∎

The smallest connected NAS graph is $K_{2,2}$ which has $t = 2$ equivalence classes under $\equiv$, each of size 2. In fact we have:

Lemma 3.9 *Let G be a connected NAS graph with t equivalence classes under $\equiv$. Then $t \geq 2$, and G has at least two equivalence classes each of size ≥ 2.*

Proof. Let G be a connected NAS graph and let $u \in V(G)$ be a vertex of maximum degree Δ, then u has a twin. So $|U| \geq 2$, and U is the first equivalence class of size ≥ 2.

Since G is connected and NAS then $|V(G)| \geq 4$, and so $\Delta \neq 0$, *i.e.*, $N_G(u) \neq \emptyset$. So let $v \in N_G(u)$ have the maximum degree amongst all vertices in $N_G(u)$. Let v^p be a parent of v, clearly $v^p \in N_G(u)$. Now $deg(v) \leq deg(v^p)$, but v has maximum degree amongst all vertices in $N_G(u)$, so $deg(v) = deg(v^p)$ and v^p is a twin of v. Hence $|V| \geq 2$, and clearly $V \neq U$, so V is another equivalence class of size ≥ 2. And also $t \geq 2$. ■

So if G is a connected NAS graph with $n \geq 4$ vertices then $t = |V(G_{\equiv})| \geq 2$. Also $t \leq n - 2$ because there are at least two labels on $V(G_{\equiv})$ that are each ≥ 2.

Now, using Theorem 3.8, we have Algorithm Connected NAS Graphs below; it constructs all connected NAS graphs G on a fixed number $n \geq 4$ of vertices, with no repeated G.

Algorithm Connected NAS Graphs Four step algorithm to construct all connected NAS graphs G on a fixed number of $n \geq 4$ vertices from all connected labelled ND graphs H on $2 \leq t \leq n - 2$ vertices.

`For each fixed` $t = 2, 3, \ldots, n-2$:

(1) `By repeated use of Algorithm ND Graphs, modified to generate connected ND graphs, list all non-isomorphic connected ND graphs` H_t `on` t `vertices.`

(2) `List all partitions` $\mathcal{P}_t$ `of` n `with` t `parts.`

(3) `For each graph` H_t `and partition` $\mathcal{P}_t = \{\ell_1, \ell_2, \ldots, \ell_t\}$ `label its` t `vertices with` $\{\ell_1, \ell_2, \ldots, \ell_t\}$ `in all possible non-label-isomorphic ways, ensuring that all parentless vertices have label` ≥ 2.

(4) `For each labelled graph` H_t `construct` $G = {H_t}^{\uparrow}$.

Example 2 As an illustration of Algorithm Connected NAS Graphs we construct all connected NAS graphs G on $n = 6$ vertices. We need only consider all connected ND graphs H_t on $t = 2$, 3, or 4 vertices. These H_t are shown in the left column of Fig. 2, suitably labelled. The last 2 such graphs have ≥ 3 parentless vertices, indicated by $\overline{p}$, each of which require a label of ≥ 2, this is not possible since the sum of all labels must equal 6.

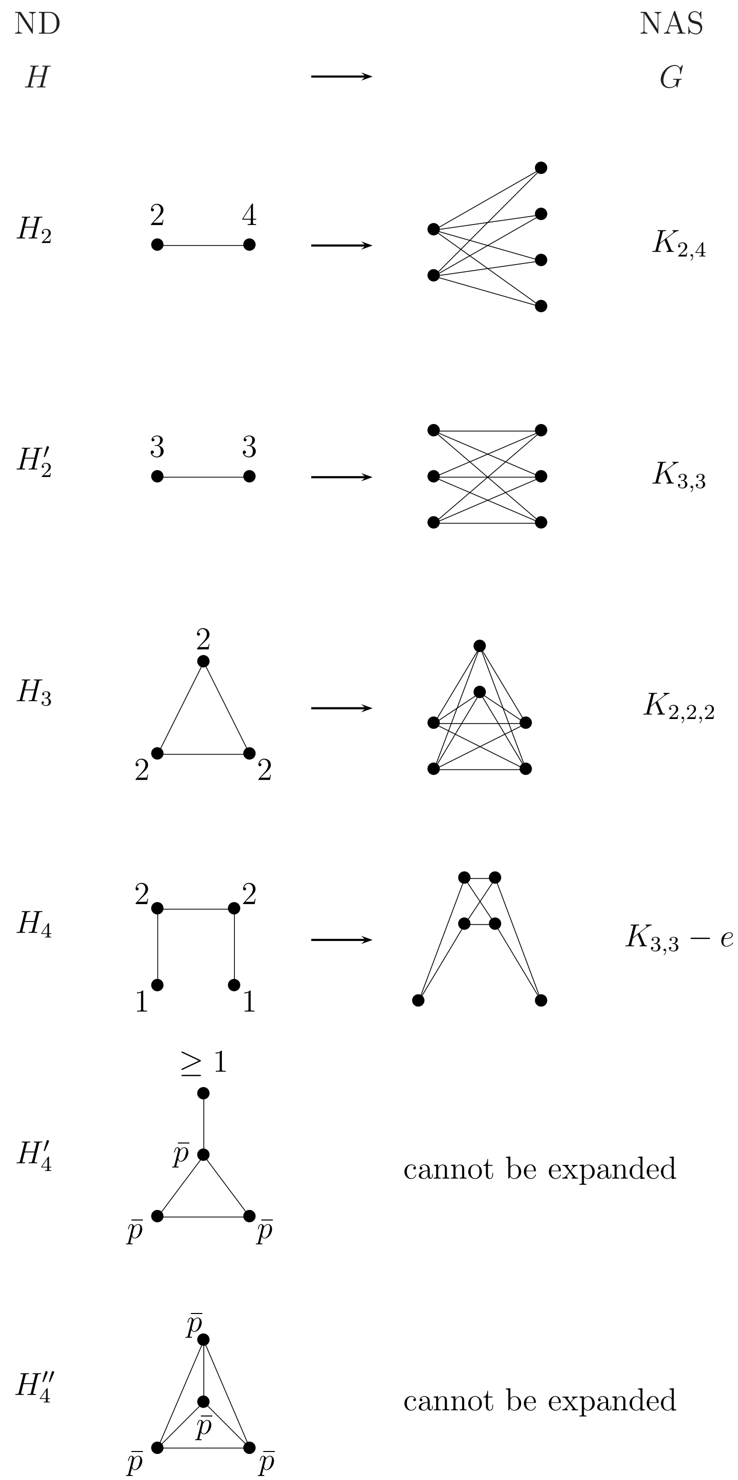


Fig. 2. All connected NAS graphs G on 6 vertices, illustrating Algorithm Connected NAS Graphs for $n = 6$

4 Classifying r-regular NAS Graphs

In [3] all connected r-regular NAS graphs were found and classified for $r = 2, 3,$ and 4, (Theorems 2.5, 2.7, and 3.5, respectively). Here we simplify the proofs of these Theorems, and extend the classification to $r = 5$ and 6.

We first require the following Lemma:

Lemma 4.1 *Let G be an arbitrary graph with equivalence relation $\equiv$. Then, for every $u \in V(G)$, the neighborhood $N_G(u)$ is a union of equivalence classes of $\equiv$.*

Proof. For any $v \in V(G)$ denote 2 arbitrary members of the equivalence class $[v]$ by v' and v''. If $v' \in N_G(u)$ then $u \in N_G(v') = N_G(v'')$, and so $v'' \in N_G(u)$ and then $[v] \subseteq N_G(u)$ since v'' is arbitrary. Hence, if $N_G(u)$ contains a part of an equivalence class then it contains the whole class, and so it is a union of whole equivalence classes, as required. ■

Let H be an arbitrary labelled graph with label $\ell(u) \geq 1$ on each vertex $u \in V(H)$, and with no isolated vertices. If distinct vertices u and $v \in V(H)$ are adjacent we write $u \sim v$. Define the *weight* of u as $wt(u) = \sum_{v \sim u} \ell(v)$. For $r \geq 1$ we say that H is *r-weight-regular* if $wt(u) = r$ for all $u \in V(H)$.

Note that $|V(H)| \geq 2$. In the remainder of this Section, when H is replaced by $G_\equiv$, we will always assume that $|V(G_\equiv)| \geq 2$, even though this may not be explicitly mentioned.

Theorem 4.2 *Let G be an arbitrary graph and let $r \geq 1$. Then G is r-regular if and only if $G_\equiv$ is r-weight-regular.*

Proof. As usual let G have t equivalence classes under $\equiv$ given by $U_1, U_2, \ldots, U_t$.

First let G be r-regular. Fix i with $1 \leq i \leq t$ and let $u \in U_i$ and $N_G(u) = \{v_1, v_2, \ldots, v_r\}$. From Lemma 4.1 above $N_G(u)$ is a union of s equivalence classes, say $N_G(u) = \bigcup_{k=1}^{s} V_k$. So $deg(u) = r = \sum_{k=1}^{s} |V_k|$. Now, in G, vertex $u \sim v$ for (every) $v \in V_k$, for all $k = 1, 2, \ldots, s$. So, in $G_\equiv$, vertex $U_i \sim V_k$, for all $k = 1, 2, \ldots, s$. Now the labels on vertices V_k are $\ell(V_k) = |V_k|$. Hence, in $G_\equiv$, we have $wt(U_i) = \sum_{k=1}^{s} \ell(V_k) = \sum_{k=1}^{s} |V_k| = r$. So, since i is arbitrary, then $G_\equiv$ is r-weight-regular.

Conversely suppose that $G_\equiv$ is r-weight-regular, let the arbitrary vertex $U \in V(G_\equiv)$ have label $\ell(U) = |U| \geq 1$. When constructing $G = (G_\equiv)^\uparrow$ from $G_\equiv$ we replace vertex U with the $\ell(U)$ vertices from $exp(U) =$

$\{u_1, u_2, \ldots, u_{\ell(U)}\}$. Then, in G, for any u_j with $j = 1, 2, \ldots, \ell(U)$, we have $deg(u_j) = \sum_{V \sim U} |V| = \sum_{V \sim U} \ell(V) = wt(U) = r$. Since $U \in V(G_{\equiv})$ is arbitrary and $u_j \in U$ is also arbitrary, then G is r-regular. ■

Theorem 4.3 *Let G be an arbitrary graph. Then G is a connected r-regular NAS graph if and only if $G_{\equiv}$ is a connected ND r-weight-regular graph, in which each label $\ell(U)$ satisfies $2 \leq \ell(U) \leq r$.*

Proof. If G is a connected r-regular NAS graph then every vertex u is a vertex of maximum degree and so has a twin, and so each $|U| \geq 2$, thus each $\ell(U) \geq 2$ in $G_{\equiv}$. Furthermore by Lemma 3.7 and Theorem 4.2 the graph $G_{\equiv}$ is a connected ND r-weight-regular graph. Noting that in a r-weight-regular graph each label must be $\leq r$, then, in $G_{\equiv}$, each label $\ell(U)$ satisfies $2 \leq \ell(U) \leq r$. The proof of the converse uses the same results. ■

Corollary 4.4 *Let G be a connected r-regular NAS graph. Then $r \geq 2$.*

Proof. From Theorem 4.3 each label on $G_{\equiv}$ is ≥ 2. Hence $r \geq 2$. ■

From Theorem 4.3, if G is a connected r-regular NAS graph then the labels on $G_{\equiv}$ must come from $L = \{2, 3, \ldots, r\}$. In the following $G_{\equiv}$ is a connected ND r-weight-regular graph. Before we start the classification of connected r-regular NAS graphs for $r = 0, 1, \ldots, 6$ we have two Observations:

(1) If label r is used on $G_{\equiv}$ then $G_{\equiv}$ must be $\overset{r}{\bullet}\!\!-\!\!\!-\!\!\overset{r}{\bullet}$. So $G = \left(\overset{r}{\bullet}\!\!-\!\!\!-\!\!\overset{r}{\bullet}\right)^{\uparrow} = K_{r,r}$, is the unique connected r-regular NAS graph in which $G_{\equiv}$ has a label r.
(2) Label $r-1$ cannot be used on $G_{\equiv}$ since, in order to make $G_{\equiv}$ r-weight-regular, we would then require label 1, but $1 \notin L$.

In the following we classify all connected r-regular NAS graphs G for a fixed $r = 0, 1, \ldots, 6$ by first finding all connected labelled ND r-weight-regular graphs $G_{\equiv}$, in which each label $\ell(U)$ satisfies $2 \leq \ell(U) \leq r$, and then using Theorem 4.3.

$\mathbf{r = 0}$ or $\mathbf{1}$ From Corollary 4.4 there are no connected r-regular NAS graphs for $r = 0$ or 1.

$\mathbf{r=2}$ $L = \{2\} = \{r\}$. So all labels on $G_{\equiv}$ must be $r = 2$. Hence, from Observation (1), $G = K_{2,2}$ is the unique connected 2-regular NAS graph.

$\mathbf{r=3}$ $L = \{2, 3\} = \{r-1, r\}$. From from Observations (1) and (2) $G = K_{3,3}$ is the unique connected 3-regular (cubic) NAS graph.

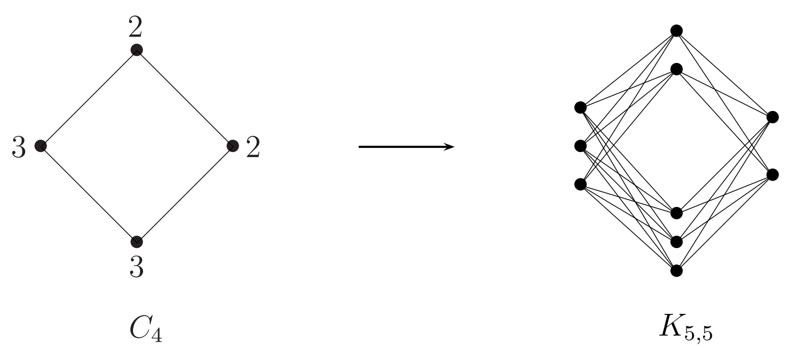
$\mathbf{r=4}$ $L = \{2, 3, 4\} = \{2, r-1, r\}$. If label $r = 4$ is used then $G = K_{4,4}$.

Clearly if only label 2 is used on $G_{\equiv}$, a connected ND $4 = \underbrace{2+2}_{2}$-weight-regular graph, then $G_{\equiv}$ must be 2-regular. So $G_{\equiv}$ is the a-cycle, C_a, where $a \geq 3$ and $a \neq 4$ since C_4 is not ND, with each vertex having label 2. But note that if we label each vertex of C_4 with 2 we get $C_4^{\uparrow} = K_{4,4}$. Hence the only connected 4-regular NAS graphs are $G = C_a^{\uparrow}$ where each vertex in C_a has label 2, for all $a \geq 3$.

Also note that G has $2a$ vertices so there are no connected 4-regular NAS graphs with an odd number of vertices, see Corollary 3.6 of [3].

$\mathbf{r=5}$ $L = \{2, 3, 4, 5\} = \{2, 3, r-1, r\}$. If label $r = 5$ is used then $G = K_{5,5}$.

Otherwise $G_{\equiv}$ is a connected ND $5 = \underbrace{2+3}_{2}$-weight-regular graph, hence, again, $G_{\equiv}$ is 2-regular. Indeed $G_{\equiv}$ is the $4a$-cycle, C_{4a}, for $a \geq 2$. The labelling on each 'block' of 4 clockwise-consecutive vertices in C_{4a} is $\{2, 2, 3, 3\}$ repeated a times until all $4a$ vertices are labelled. The case $a = 1$ gives C_4 which is not ND, however C_4 labelled with $\{2, 2, 3, 3\}$ gives $C_4^{\uparrow} = K_{5,5}$. So the only connected 5-regular NAS graphs are $G = C_{4a}^{\uparrow}$ with the above labelling, for all $a \geq 1$. The example below shows $a = 1$.



$\mathbf{r=6}$ $L = \{2,3,4,5,6\} = \{2,3,4,r-1,r\}$. If label $r=6$ is used then we get $K_{6,6}$.

Since label 5 cannot be used on $G_\equiv$ we can only use labels from $L' = \{2,3,4\}$. So let $G_\equiv$ be a connected ND 6-weight-regular graph with labels from $L' = \{2,3,4\}$. Let $U \in V(G_\equiv)$. Then U has label $\ell(U) \in L'$, and since $wt(U) = 6$ there are only three possibilities for labelling U and its neighbors:

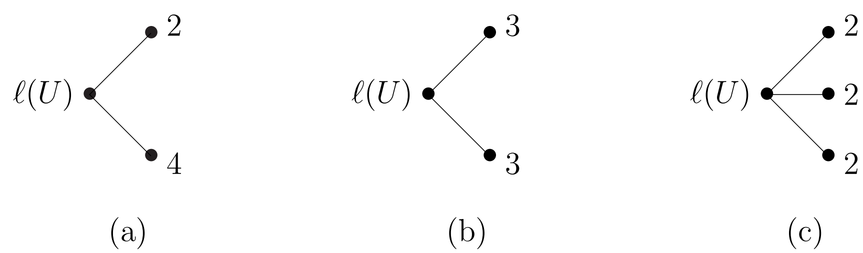


Fig. 3. Three possibilities for labelling U and its neighbors

Let a $[2,3]$-*regular* graph be a graph in which every vertex has degree 2 or 3, and with at least one vertex of degree 2 and at least one of degree 3. Then from Figs. 3(a), (b), and (c) any $G_\equiv$ must be 2-regular, or cubic, or $[2,3]$-regular.

In the following when we say that $G_\equiv$ is labelled with set L'', then *every* label in L'' must be used at least once. We consider L'' where $\emptyset \neq L'' \subseteq L'$.

The use of label $3 \in L''$ is restricted by the following Lemma. Let $d(U,V)$ be the length of a shortest path between vertices U and V in $G_\equiv$.

Lemma 4.5 *Let $G_\equiv$ be labelled with the set L'' where $3 \in L''$, and let U and V be distinct vertices of $G_\equiv$ with $UV \in E(G_\equiv)$.*

(i) Then either $\ell(U) = 3$ or $\ell(V) = 3$ (or both).

(ii) If both $\ell(U) = \ell(V) = 3$. Then $L'' = \{3\}$.

Proof. Recall that $G_\equiv$ is connected.
(i) Since $3 \in L''$ then 3 must be used as a label in $G_\equiv$, so let $W \in V(G_\equiv)$ have $\ell(W) = 3$. If $W = U$ or V then we are finished. Otherwise every second vertex on any path starting at W must have label 3, see Fig. 3(b). So, if $d(W,U)$ is even, then $\ell(U) = 3$. Or if $d(W,U)$ is odd then there is a path of even length $d(W,U) \pm 1$ from W to V, hence $\ell(V) = 3$.
(ii) Let X be any vertex in $V(G_\equiv)$. Then either there is a path of even length from U to X or from V to X. Hence $\ell(X) = 3$. So every vertex has

label 3, *i.e.*, $L'' = \{3\}$. ■

Now we are ready to consider the seven cases for L'' where $\emptyset \neq L'' \subseteq \{2, 3, 4\}$. We will determine all possible $G_\equiv$, connected ND 6-weight-regular graphs. Then all G, connected 6-regular NAS graphs, are given by $G = (G_\equiv)^\uparrow$.

$\mathbf{L''} = \{\mathbf{2}\}$ $G_\equiv$ is any connected ND cubic graph, with label 2 on every vertex.

$\mathbf{L''} = \{\mathbf{3}\}$ $G_\equiv$ is a connected ND 2-regular graph. So $G_\equiv = C_a$, where $a \geq 3$, and $a \neq 4$, with each vertex having label 3. (As usual note that a labelled $C_4^\uparrow = K_{6,6}$.)

$\mathbf{L''} = \{\mathbf{4}\}$ or $\{\mathbf{3}, \mathbf{4}\}$ No $G_\equiv$ are possible since a label of 2 is needed with the label of 4 to create a vertex with weight 6.

$\mathbf{L''} = \{\mathbf{2}, \mathbf{3}\}$ For this case we show that there is a one-to-one correspondence between the set of $G_\equiv$ and the set of simple connected cubic graphs S.

Here $G_\equiv$ is a $[2, 3]$-regular connected ND 6-weight-regular graph labelled with $L'' = \{2, 3\}$. Let $U \in V(G_\equiv)$. Suppose that $\ell(U) = 2$ but that U has degree 3. Then this contradicts Fig. 3(c) and Lemma 4.5(i), hence U has degree 2. Conversely, let vertex U have degree 2, so, from Fig. 3(b) and Lemma 4.5(ii), we have $\ell(U) = 2$. So a vertex in $G_\equiv$ has label 2 if and only if it has degree 2, and has label 3 if and only if it has degree 3.

Let $UV \in E(G_\equiv)$ be arbitrary. Then, from Lemma 4.5(i) and (ii), without loss of generality let $\ell(U) = 2$ and $\ell(V) = 3$. So, for any edge in $G_\equiv$, one end-vertex has label 2 and the other has label 3. Hence $G_\equiv$ is a bipartite graph with all vertices U with $\ell(U) = 2$ in the first part, call this set of vertices $\mathcal{U}$; and all vertices V with $\ell(V) = 3$ in the second part, call this set $\mathcal{V}$.

From above each vertex in $\mathcal{U}$ has degree 2 and each vertex in $\mathcal{V}$ has degree 3. Counting edges gives $2|\mathcal{U}| = 3|\mathcal{V}|$. So $|\mathcal{V}|$ is even, set $|\mathcal{V}| = 2a$, then $|\mathcal{U}| = 3a$, where $a \geq 2$. (If $a = 1$ then $G_\equiv = K_{3,2}$ which is not ND; but, as usual, a labelled $K_{3,2}^\uparrow = K_{6,6}$.)

Let $\mathcal{U} = \{U_1, U_2, \ldots, U_{3a}\}$ and $\mathcal{V} = \{V_1, V_2, \ldots, V_{2a}\}$. Now construct a graph S with vertex-set $V(S) = \mathcal{V}$, and with edge $V_jV_{j'}$ ($j \neq j'$) whenever there is a vertex $U_i \in \mathcal{U}$ with $N_{G_\equiv}(U_i) = \{V_j, V_{j'}\}$. So S has $2a$ vertices and $3a$ edges. Now S has no loops, and also no multi-edges because $G_\equiv$ is ND, so S is simple. And S is cubic because the degree of any V_j in $G_\equiv$ is 3. Finally S is connected because $G_\equiv$ is connected. Thus S is a simple connected cubic graph on $2a$ vertices.

Conversely, let S be a simple connected cubic graph on the $2a$ vertices $\mathcal{V} = \{V_1, V_2, \ldots, V_{2a}\}$, where $a \geq 2$. Then S has $3a$ edges $E(S) = \{e_1, e_2, \ldots, e_{3a}\}$. Put a new vertex U_i on edge e_i for each $i = 1, 2, \ldots, 3a$. Let $\mathcal{B}$ be the bipartite graph with vertex sets $\mathcal{U} = \{U_1, U_2, \ldots, U_{3a}\}$ as its first part and $\mathcal{V}$ as its second part. Then $\mathcal{B}$ is $[2, 3]$-regular, and, because S is connected, then $\mathcal{B}$ is connected. And clearly $\mathcal{B}$ is ND. Now label each vertex U_i with 2, and each vertex V_j with 3. Then $\mathcal{B}$ is a 6-weight-regular graph. Thus $\mathcal{B}$ is a $[2, 3]$-regular connected ND 6-weight-regular graph, *i.e.*, $\mathcal{B}$ is an example of a $G_{\equiv}$. We denote $\mathcal{B}$ by $Bis(S)$, because we have 'bisected' each edge in S to form $\mathcal{B}$.

Thus there is a one-to-one correspondence between the set of $G_{\equiv}$ and the set of simple connected cubic graphs S.

$\mathbf{L}'' = \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$ Here we show that the structure of $G_{\equiv}$ is quite restricted, but show that infinitely many such $G_{\equiv}$ exist.

By a similar argument in the above $L'' = \{2, 3\}$ case, $G_{\equiv}$ is a bipartite graph. Its first part consists of all vertices of degree 2 and labels 2 or 4; let there be $a \geq 1$ vertices $\mathcal{U}$ with label 2 and $b \geq 1$ vertices $\mathcal{U}'$ with label 4. Its the second part consists of vertices with label 3; let there be $c \geq 0$ vertices $\mathcal{V}$ with degree 3 and $d \geq 0$ vertices $\mathcal{V}'$ with degree 2, where $c + d \geq 1$. See Fig. 4.

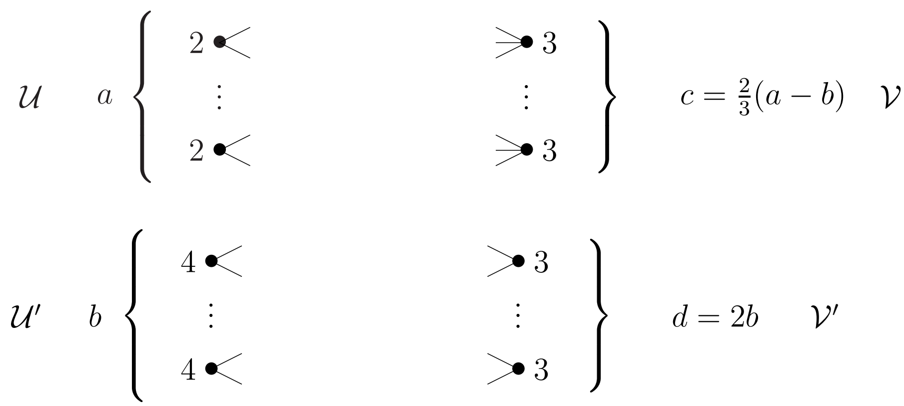


Fig. 4. The graph $G_{\equiv}(a, b)$

We show that $d = 2b$ and $c = \frac{2}{3}(a - b)$. Consider the $2b$ edges incident to the b vertices in $\mathcal{U}'$. From Fig. 3 each of these $2b$ edges is incident to a vertex in $\mathcal{V}'$. Now a vertex in $\mathcal{V}'$ cannot be adjacent to two vertices in $\mathcal{U}'$ since such a vertex would then have weight 8. So these $2b$ edges must be incident to $2b$ distinct vertices in $\mathcal{V}'$, hence $2b \leq d$. If $d > 2b$ then $\mathcal{V}'$ contains a vertex that is not adjacent to any vertex in $\mathcal{U}'$, so it must be adjacent to two vertices in $\mathcal{U}$ and so have weight 4, a contradiction. Hence $d \leq 2b$, and so $d = 2b$. Then counting edges in two different ways gives $c = \frac{2}{3}(a-b)$, so $a \geq b \geq 1$ and $a \equiv b \pmod 3$.

To summarize: $G_{\equiv}$ is a connected ND bipartite graph. The first part consists of $a \geq 1$ vertices with degree 2 and label 2 ($\mathcal{U}$), and $b \geq 1$ vertices with degree 2 and label 4 ($\mathcal{U}'$). The second part consists of $\frac{2}{3}(a-b) \geq 0$ vertices with degree 3 and label 3 ($\mathcal{V}$), and $2b \geq 2$ vertices with degree 2 and label 3 ($\mathcal{V}'$), where $a \geq b \geq 1$ and $a \equiv b \pmod 3$. Furthermore, the $2b$ edges incident to the b vertices in $\mathcal{U}'$ are incident to the $2b$ vertices in $\mathcal{V}'$, in a one-to-one fashion. We denote such a graph by $G_{\equiv}(a,b)$.

If $a = b$ then $G_{\equiv}(a,a)$ is 2-regular. So $G_{\equiv}(a,a) = C_{4a}$ for $a \geq 2$ with labelling $\{2,3,4,3\}$ repeated a times around the cycle.

If $a > b$ then $G_{\equiv}(a,b)$ is $[2,3]$-regular. It doesn't seem possible to explicitly describe all $G_{\equiv}(a,b)$; we can, however, show that a $G_{\equiv}(a,b)$ exists for all ordered pairs (a,b) with $a > b \geq 1$ and $a \equiv b \pmod 3$, except for $(a,b) = (4,1)$; see Theorem 4.9.

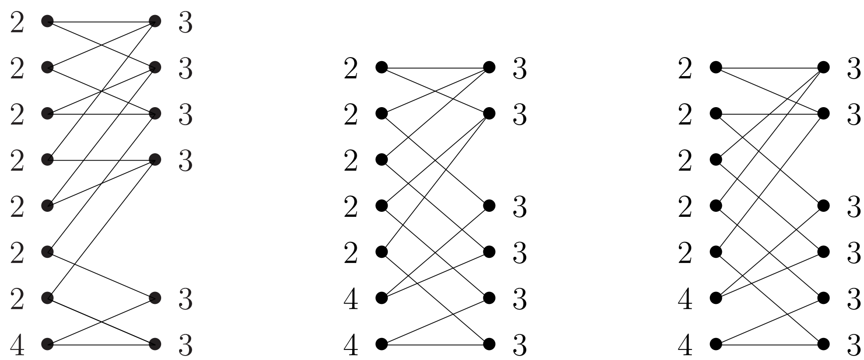
First we deal with the pairs $(a,b) = (4,1)$, $(7,1)$, and $(5,2)$. We omit the proof of the following Lemma, it involves straightforward exhaustive checking.

Lemma 4.6

(i) There does not exist a $G_{\equiv}(4,1)$*.*

(ii) There exists a unique $G_{\equiv}(7,1)$ *up to label-isomorphism.*

(iii) There exist two non-label-isomorphic $G_{\equiv}(5,2)$*'s, one with girth* 6 *and one with girth* 8*.* ■



$G_{\equiv}(7,1)$ unique | $G_{\equiv}(5,2)$ girth 6 | $G_{\equiv}(5,2)$ girth 8

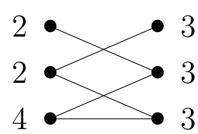
Fig. 5. Three small $G_{\equiv}(a,b)$

Lemma 4.7 *Suppose that there exists a $G_{\equiv}(a,b)$ with $a > b \geq 1$. Then*

(i) there exists a $G_{\equiv}(a+1, b+1)$,

(ii) there exists a $G_{\equiv}(a+3, b)$.

Proof. (i) Clearly any $G_{\equiv}(a,b)$ must contain an edge . Delete this edge and replace it with:

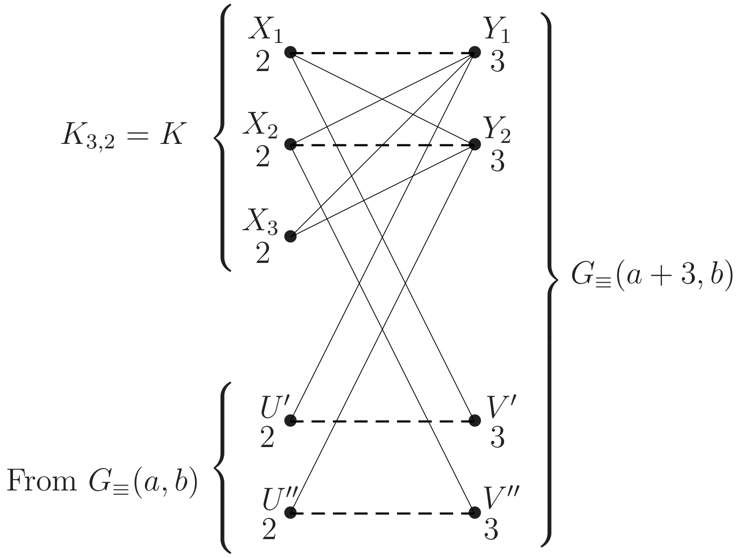


We now have a $G_{\equiv}(a+1, b+1)$.
(ii) In $G_{\equiv}(a,b)$, as usual, let $\mathcal{U} = \{U_1, U_2, \ldots, U_a\}$ where, from Lemma 4.6, we have $a \geq 5$. And let $\mathcal{V} \cup \mathcal{V}' = \{V_1, V_2, \ldots, V_{\frac{2}{3}(a+2b)}\}$, where the $\frac{2}{3}(a-b) \geq 2$ vertices from $\mathcal{V}$ are listed first; we have $\frac{2}{3}(a+2b) \geq 6$.

We first show that we can find two edges $U'\,V'$ and $U''\,V''$ where $U', U'' \in \mathcal{U}$ are distinct and $V', V'' \in \mathcal{V} \cup \mathcal{V}'$ are distinct, such that $U'\,V''$ and $U''\,V'$ are non-edges.

Without loss of generality let U_1V_1 be an edge of $G_{\equiv}(a,b)$, and choose U_1V_1 as the first edge $U'V'$, *i.e.*, let $U' = U_1$ and $V' = V_1$. Now $deg(U') = 2$, let $N_{G_{\equiv}(a,b)}(U') = \{V', V(1)\}$ for some $V(1) \in \mathcal{V} \cup \mathcal{V}' \backslash \{V'\}$. Also $deg(V') = 3$ so, without loss of generality, let $N_{G_{\equiv}(a,b)}(V') = \{U', U_2, U_3\}$. Now consider U_4 and let $U'' = U_4$. Let $N_{G_{\equiv}(a,b)}(U'') = \{V(2), V(3)\}$. At least one of $V(2)$ or $V(3)$ is distinct from $V(1)$, say $V(2) \neq V(1)$, (and both are distinct from V'). So let $V'' = V(2)$. Then $U''V''$ is an edge of $G_{\equiv}(a,b)$, and $U'V''$ and $U''V'$ are non-edges. And $\ell(U') = \ell(U'') = 2$ and $\ell(V') = \ell(V'') = 3$.

Now let K be a copy of $K_{3,2}$ with vertices $\{X_1, X_2, X_3\}$ in the first part, each with label 2; and vertices $\{Y_1, Y_2\}$ in the second part, each with label 3. In $K \cup G_{\equiv}(a,b)$ remove the 2 edges $\{X_1Y_1, U'V'\}$ and add the 2 edges $\{X_1V', Y_1U'\}$, then remove the 2 edges $\{X_2Y_2, U''V''\}$ and add the 2 edges $\{X_2V'', Y_2U''\}$. See below where the removed edges are indicated by dashed lines.



Then careful checking shows that this new graph is a $G_{\equiv}(a+3,b)$. ∎

Lemmas 4.6(ii) and 4.7(ii) give:

Lemma 4.8 *There exists a* $G_{\equiv}(a,1)$ *for all* $a \geq 7$ *with* $a \equiv 1 \ (mod\ 3)$. ∎

Theorem 4.9 *There exists a* $G_{\equiv}(a,b)$ *for all pairs* (a,b) *with* $a > b \geq 1$ *and* $a \equiv b \ (mod\ 3)$, *except for* $(a,b) = (4,1)$.

Proof. First suppose that $a - b = 3$. Since a $G_{\equiv}(4, 1)$ does not exist then $a \geq 5$. Now start with a $G_{\equiv}(5, 2)$ and apply Lemma 4.7(i) a total of $a - 5$ times to reach a $G_{\equiv}(5 + (a - 5), 2 + (a - 5)) = G_{\equiv}(a, a - 3) = G_{\equiv}(a, b)$.

If $a - b = 6, 9, 12, \ldots$. Then start with a $G_{\equiv}(a - b + 1, 1)$ which exists by Lemma 4.8 and apply Lemma 4.7(i) a total of $b - 1$ times to reach a $G_{\equiv}(a, b)$. ■

Remark In the four cases $L'' = \{2\}, \{3\}, \{2, 3\}$ or $\{2, 3, 4\}$ that yield a $G_{\equiv}$ the labelling on $G_{\equiv}$ is unique. That is, for any (unlabelled) $G_{\equiv}$ constructed by the method shown for each case, there is a unique way, up to label-isomorphism, to label its vertices with L'' to produce a 6-weight-regular graph, (see step (3) of Algorithm Connected NAS Graphs).

$\mathbf{L'' = \{2, 4\}}$ In this final case we call a vertex with degree 3 a *cubic* vertex.

If $G_{\equiv}$ is 2-regular then $G_{\equiv} = C_{4a}$ for $a \geq 2$ with labelling $\{2, 2, 4, 4\}$ repeated a times around the cycle.

So let $G_{\equiv}$ be a $[2, 3]$-regular connected ND 6-weight-regular graph with $a \geq 1$ vertices of degree 2 and $b \geq 2$ cubic vertices – so b is even – that has been labelled with $L'' = \{2, 4\}$; denote such a graph by $G_{\equiv}\{a, b\}$. Similar to the previous case with $L'' = \{2, 3, 4\}$ an exact classification of $G_{\equiv}\{a, b\}$ doesn't seem possible. We can, however, determine all pairs $\{a, b\}$ for which there exists a $G_{\equiv}\{a, b\}$; see Theorem 4.15.

First we need the following three results.

Lemma 4.10 *In a $G_{\equiv}\{a, b\}$ the number of vertices of degree 2 equals the sum of the degrees of all vertices with label 4,* i.e.,

$$a = \sum_{\ell(U)=4} deg(U). \tag{1}$$

Proof. If a vertex has degree 2 then it must have a neighbor with label 4, see Fig. 3(a). Conversely if a vertex has label 4 then all of its neighbors must have degree 2. Hence the result. ■

Now the following Corollary is immediate.

Corollary 4.11 *In a* $G_{\equiv}\{a,b\}$

(i) if a *is even then there are an even number of cubic vertices with label* 4*,*

(ii) if a *is odd then there are an odd number of cubic vertices with label* 4. ■

The next result concerns vertices in $G_{\equiv}\{a,b\}$ with degree 2 and label 4.

Lemma 4.12 *In a* $G_{\equiv}\{a,b\}$

(i) a vertex with degree 2 *and label* 4 *is adjacent to exactly one other vertex with degree* 2 *and label* 4*,*

(ii) vertices with degree 2 *and label* 4 *occur in adjacent pairs,*

(iii) there are an even number of vertices with degree 2 *and label* 4.

Proof. (i) Let U have degree 2 and label 4 and let U_1 and U_2 be the 2 neighbors of U, with $\ell(U_1) = 4$ and $\ell(U_2) = 2$. Then $deg(U_1) = 2$, and U_1 is the required vertex which is adjacent to U with degree 2 and label 4.
(ii) Now the other neighbor of U_1 has label 2. So adjacent vertices U and U_1 have as neighbors, excepting each other, a vertex with label 2. Thus U and U_1 form an adjacent pair of vertices each with degree 2 and label 4.
(iii) This now follows from (ii). ■

These three results are now used in the following non-existence proofs, given in order of increasing a.

Theorem 4.13 *Graphs* $G_{\equiv}\{a,b\}$ *do not exist as indicated below:*

(i) $a = 1$*: there does not exist a* $G_{\equiv}\{1,b\}$ *for all even* $b \geq 2$*,*

(ii) $a = 2$*: there does not exist a* $G_{\equiv}\{2,b\}$ *for all even* $b \geq 2$*,*

(iii) $a = 3$*: there does not exist a* $G_{\equiv}\{3,2\}$*,*

(iv) $a = 4$*: there does not exist a* $G_{\equiv}\{4,2\}$*,*

(v) $a = 5$*: there does not exist a* $G_{\equiv}\{5,b\}$ *for all even* $b \geq 2$*,*

(vi) $a = 4s+1$*, for all* $s \geq 2$*: there does not exist a* $G_{\equiv}\{a,2\}$.

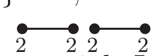
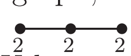
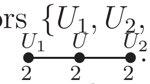
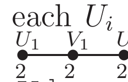
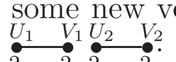
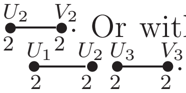
Proof. (i) In a $G_{\equiv}\{a,b\}$ we have $a = \sum_{\ell(U)=4} deg(U)$ from Equation (1). So in a $G_{\equiv}\{1,b\}$ we have $1 = \sum_{\ell(U)=4} deg(U) \geq 2$ since there is at least one vertex with degree 2 or 3 with label 4, a contradiction.
(ii) Similarly, in a $G_{\equiv}\{2,b\}$ we have $2 = \sum_{\ell(U)=4} deg(U)$ so there is exactly 1 vertex with degree 2 and label 4, a contradiction to Lemma 4.12(iii).
(iii) A $G_{\equiv}\{3,2\}$ has degree sequence $2^3 3^2$. There are only two such graphs with this degree sequence, G43 and G44 on p. 8 of [4]. But graph G43 cannot be labelled with L'' to be 6-weight-regular, and G44 is not ND.
(iv) Similar to (iii) using p. 10 of [4], see the graphs G127, G128, G129, and G130.
(v) From Equation (1) a $G_{\equiv}\{5,b\}$ has exactly 1 cubic vertex with label 4, and so 1 vertex with degree 2 and label 4, again a contradiction to Lemma 4.12(iii).
(vi) Since $a = 4s+1$ is odd then, by Equation (1) and Corollary 4.11(ii), a $G_{\equiv}\{a,2\}$ has exactly 1 cubic vertex with label 4. Then Equation (1) becomes $4s+1 = 3 + \sum_{\substack{\ell(U)=4 \\ deg(U)=2}} deg(U)$, and so $G_{\equiv}\{a,2\}$ has $2s-1$, an odd number, of vertices with degree 2 and label 4, again a contradiction. ∎

Now we show the existence of every $G_{\equiv}\{a,b\}$ not mentioned in Theorem 4.13. We need:

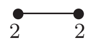
Lemma 4.14 *Suppose that there exists a $G_{\equiv}\{a,b\}$ with $a \geq 3$ and even $b \geq 2$. Then*

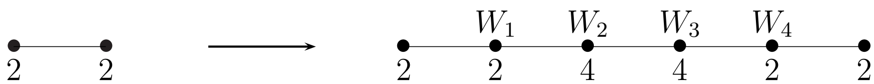
(i) there exists a $G_{\equiv}\{a+4,b\}$,

(ii) there exists a $G_{\equiv}\{a,b+2\}$.

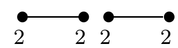
Proof. We first show that every $G_{\equiv}\{a,b\}$ has, as a subgraph, at least one of the two following edge configurations:  or , the degrees of the vertices involved are immaterial. Let U be a cubic vertex in $G_{\equiv}\{a,b\}$ with neighbors $\{U_1,U_2,U_3\}$, so $\ell(U_1)=\ell(U_2)=\ell(U_3)=2$. If $\ell(U)=2$ then we have . If $\ell(U)=4$ then each U_i has degree 2 and label 2. So each U_i is adjacent to one vertex with label 2. If there are no edges in the induced graph $G_{\equiv}\{a,b\}[U_1,U_2,U_3]$, then each U_i must be adjacent to some new vertex V_i. If $V_1 = V_2$ then we have ; or else we have . Or with 1 edge in $G_{\equiv}\{a,b\}[U_1,U_2,U_3]$, say edge U_1U_2, we must have . Now 2 or 3 edges in $G_{\equiv}\{a,b\}[U_1,U_2,U_3]$

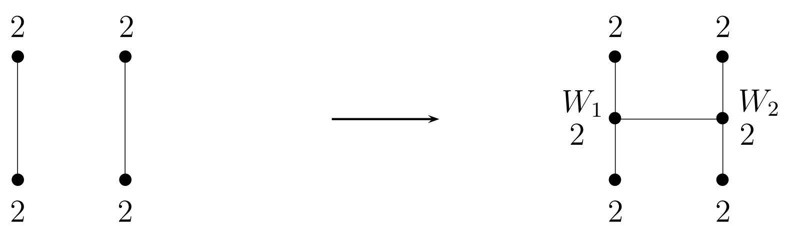
are not possible. So, in every case, we have one of the two required edge configurations.

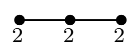
(i) In any $G_{\equiv}\{a, b\}$ replace any edge 
with the path on 6 vertices, as shown below

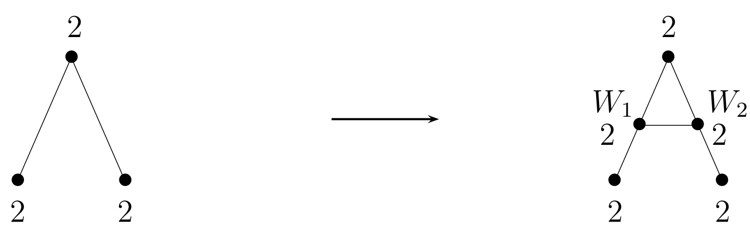


to produce a $G_{\equiv}\{a+4, b\}$. The vertices $\{W_1, W_2, W_3, W_4\}$ are the 4 new vertices with degree 2.

(ii) In our $G_{\equiv}\{a, b\}$ amend the edge configuration  as shown below



or the edge configuration  as shown below



to produce a $G_{\equiv}\{a, b+2\}$. The vertices $\{W_1, W_2\}$ are the 2 new cubic vertices. ∎

Now we give the main existence result:

Theorem 4.15 *Graphs $G_{\equiv}\{a,b\}$ exist as indicated below:*

(i) $a \equiv 0$ (mod 4),

> *1. $a = 4$: there exists a $G_{\equiv}\{4,b\}$ for all even $b \geq 4$,*
>
> *2. $a \geq 8$: there exists a $G_{\equiv}\{a,b\}$ for all even $b \geq 2$;*

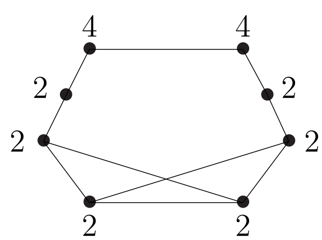
(ii) $a \equiv 1$ (mod 4), $a \geq 9$: there exists a $G_{\equiv}\{a,b\}$ for all even $b \geq 4$;

(iii) $a \equiv 2$ (mod 4), $a \geq 6$: there exists a $G_{\equiv}\{a,b\}$ for all even $b \geq 2$;

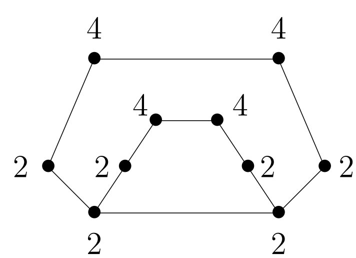
(iv) $a \equiv 3$ (mod 4),

> *1. $a = 3$: there exists a $G_{\equiv}\{3,b\}$ for all even $b \geq 4$,*
>
> *2. $a \geq 7$: there exists a $G_{\equiv}\{a,b\}$ for all even $b \geq 2$.*

Proof. (i) 1. $a \equiv 0$ (mod 4), $a = 4$: By Theorem 4.13(iv) a $G_{\equiv}\{4,2\}$ does not exist. A $G_{\equiv}\{4,4\}$ is shown on the left below:



$G_{\equiv}\{4,4\}$
unique



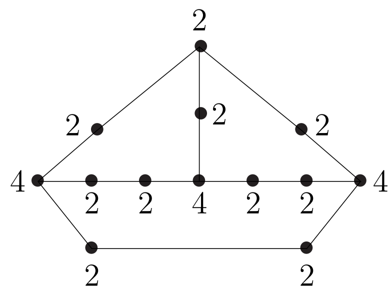
$G_{\equiv}\{8,2\}$

Then using this $G_{\equiv}\{4,4\}$ as a starter we can construct a $G_{\equiv}\{4,b\}$ for all even $b \geq 4$ by applying Lemma 4.14(ii) a total of $(b-4)/2$ times.

2. $a \equiv 0$ (mod 4), $a \geq 8$: A $G_{\equiv}\{8,2\}$ is shown on the right above, and using this as a starter we can construct a $G_{\equiv}\{a,b\}$ for all $a \geq 8$ and even $b \geq 2$ by applying Lemma 4.14(i) $(a-8)/4$ times, followed by Lemma 4.14(ii) $(b-2)/2$ times.

(ii) $a \equiv 1$ (mod 4), $a \geq 9$: By Theorem 4.13(i) and (v) graphs $G_{\equiv}\{1,b\}$ and $G_{\equiv}\{5,b\}$ for all even $b \geq 2$ do not exist; and by (vi) a $G_{\equiv}\{9,2\}$ does not exist. A $G_{\equiv}\{9,4\}$ is:

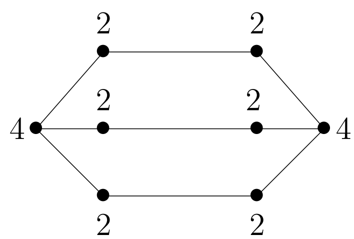
$G_{\equiv}\{9,4\}$



And using this $G_{\equiv}\{9,4\}$ as a starter we can construct a $G_{\equiv}\{a,b\}$ for all $a \geq 9$ and even $b \geq 4$ by using Lemma 4.14(i) $(a-9)/4$ times, followed by Lemma 4.14(ii) $(b-4)/2$ times.

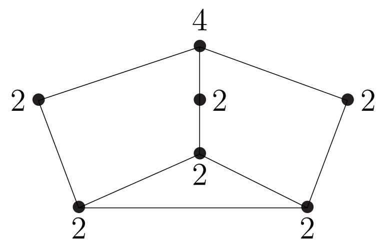
(iii) $a \equiv 2 \pmod 4$, $a \geq 6$: By Theorem 4.13(ii) graphs $G_{\equiv}\{2,b\}$ for all even $b \geq 2$ do not exist; a $G_{\equiv}\{6,2\}$ is:

$G_{\equiv}\{6,2\}$

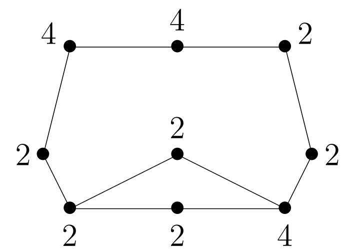


And using this $G_{\equiv}\{6,2\}$ as a starter we can construct a $G_{\equiv}\{a,b\}$ for all $a \geq 6$ and even $b \geq 2$ from $(a-6)/4$ iterations of Lemma 4.14(i), followed by $(b-2)/2$ iterations of Lemma 4.14(ii).

(iv) 1. $a \equiv 3 \pmod 4$, $a = 3$: By Theorem 4.13(iii) a $G_{\equiv}\{3,2\}$ does not exist. A $G_{\equiv}\{3,4\}$ is shown on the left below:



$G_{\equiv}\{3,4\}$
unique



$G_{\equiv}\{7,2\}$

Then using this $G_{\equiv}\{3,4\}$ as a starter we can construct a $G_{\equiv}\{3,b\}$ for all even $b \geq 4$ by applying Lemma 4.14(ii) $(b-4)/2$ times.

2. $a \equiv 3 \pmod 4$, $a \geq 7$: A $G_{\equiv}\{7,2\}$ is shown on the right above, and using this we can construct a $G_{\equiv}\{a,b\}$ for all $a \geq 7$ and even $b \geq 2$ by applying Lemma 4.14(i) $(a-7)/4$ times, followed by Lemma 4.14(ii) $(b-2)/2$ times.

■

Finally, we summarize the $r = 6$ case:

Theorem 4.16 *Let G be a NAS graph. Then $G = (G_{\equiv})^{\uparrow}$ is connected and 6-regular if and only if $G_{\equiv}$ is*

(i) any connected ND cubic graph, with label 2 on every vertex, or

(ii) the cycle C_a for $a \geq 3$, with label 3 on every vertex, or

(iii) any graph $Bis(S)$ where S is a simple connected cubic graph, with label 2 on the new vertices and label 3 on the (old) vertices of S, or

(iv) 1. the cycle C_{4a} for $a \geq 2$, with labelling $\{2,3,4,3\}$ repeated cyclically on the vertices, or

2. any graph $G_{\equiv}(a,b)$ for all pairs (a,b) with $a > b \geq 1$ and $a \equiv b \pmod 3$, except for $(a,b) = (4,1)$, labelled as in Fig. 4, or

(v) 1. the cycle C_{4a} for $a \geq 2$, with labelling $\{2,2,4,4\}$ repeated cyclically on the vertices, or

2. any graph $G_{\equiv}\{a,b\}$ from Theorem 4.15. ■

5 Extensions and Conclusions

Recall from Section 1 and [3] that if H is an arbitrary graph with t vertices and $T_1, T_2, \ldots, T_t$ are also arbitrary then the graph $G = I(T_1, T_2, \ldots, T_t : H)$ is obtained from H by replacing vertex i with a copy of T_i for each $i = 1, 2, \ldots, t$, and then if $ij \in E(H)$ joining T_i to T_j.

Our Theorem 3.2 applied to NAS graphs gives:

Theorem 5.1 *Let G be a NAS graph with reduced graph $G_\equiv$ on t vertices with labels $\ell_1, \ell_2, \ldots, \ell_t$. Then $G = I(N_{\ell_1}, N_{\ell_2}, \ldots, N_{\ell_t} : G_\equiv)$.* ■

The main result of Section 3 of [3] (Equation (1)) is our Theorem 5.1 applied to *regular* NAS graphs. Thus we have extended this result from regular NAS graphs to arbitrary NAS graphs G. We then used this extension to give two Algorithms for constructing all NAS graphs and all connected NAS graphs G from ND graphs H. The concept of a r-weight-regular-graph was then introduced to extend the classification of connected r-regular NAS graphs to $r = 5$ and 6.

Now we extend Theorem 2.11 of [3], quoted here for convenience with slight notational changes:

> *Let H be an arbitrary graph with t vertices and let $T_1, T_2, \ldots, T_t$ be a collection of t NAS graphs. Then $G = I(T_1, T_2, \ldots, T_t : H)$ is a NAS graph.*

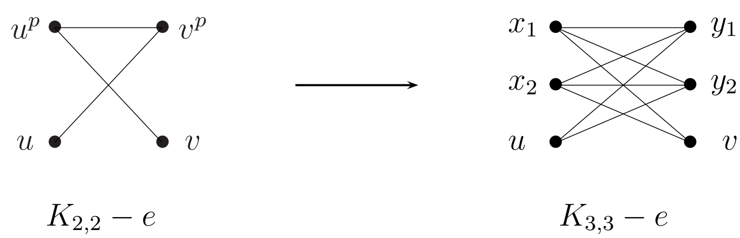
In our Section 3 we distinguished between vertices in $G_\equiv$ with and without a parent. We extend Theorem 2.11 of [3] by noting that only a parentless vertex in H need be replaced with a NAS graph to obtain a NAS graph G. Vertices with a parent in H need not be replaced, or, equivalently, they can be 'replaced' with the non-NAS graph N_1 or a NAS graph T.

Theorem 5.2 *Let H be an arbitrary graph with t vertices $\{u_1, u_2, \ldots, u_t\}$ and let $T_1, T_2, \ldots, T_t$ be a collection of t NAS graphs. Then the graph $G = I(T_1, T_2, \ldots, T_t : H)$ in which, for each $i = 1, 2, \ldots, t$, if u_i is parentless in H then it is replaced by T_i, or if u_i has a parent in H then either it is replaced by N_1 or by T_i, is a NAS graph.*

Proof. The proof follows the proof of Theorem 2.11 of [3] except when u_i has a parent u_j in H and is replaced by N_1 in forming G. Then $N_H(u_i) \subseteq$

$N_H(u_j)$ and so $N_G(u_i) \subseteq N_G(x)$ for any $x \in V(T_j)$, (where $V(T_j) = \{u_j\}$ if u_j itself is also replaced by N_1). ∎

As an illustration of this extension of Theorem 2.11 of [3] we consider the 6 vertex NAS graph $K_{3,3} - e$. Using Theorem 2.11 of [3] it is not possible to construct $K_{3,3} - e$ from a graph with fewer vertices except in the trivial manner: $K_{3,3} - e = I(K_{3,3} - e : N_1)$, from the one vertex graph N_1. Now using Theorem 5.2 above we can construct $K_{3,3} - e$ from the 4 vertex graph $K_{2,2} - e$. Let $V(K_{2,2} - e) = \{u, u^p, v, v^p\}$ in this order, shown below. Then $K_{3,3} - e = I(N_1, N_2, N_1, N_2 : K_{2,2} - e)$. So the parentless vertices u^p and v^p are each replaced by the NAS graph N_2, and the vertices u and v with parents are each replaced by N_1.



Further we note that $K_{2,2} - e$ suitably labelled with $\{1, 2, 1, 2\}$ is $(K_{3,3} - e)_{\equiv}$, so this example is also an illustration of Theorem 5.1 above, *i.e.*, $K_{3,3} - e = I(N_1, N_2, N_1, N_2 : (K_{3,3} - e)_{\equiv})$.

Acknowledgement: We thank the referees for some helpful comments.

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