K_p-Removable Sequences of Graphs

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Abstract

Let $\{G_{pn} \,|\, n \geq 1\} = \{G_{p1}, G_{p2}, G_{p3}, \ldots\}$ be a countable sequence of simple graphs, where G_{pn} has pn vertices. This sequence is called K_p-removable if $G_{p1} = K_p$, and $G_{pn} - K_p = G_{p(n-1)}$ for every $n \geq 2$ and for every K_p in G_{pn}. We give a general construction of such sequences. We specialize to sequences in which each G_{pn} is regular; these are called regular (K_p, λ)-removable sequences, λ is a fixed number, $0 \leq \lambda \leq p$, referring to the fact that G_{pn} is $(\lambda(n-1)+p-1)$-regular. We classify regular $(K_p, 0)$-, $(K_p, p-1)$-, and (K_p, p)-removable sequences as the sequences $\{nK_p \,|\, n \geq 1\}$, $\{K_{p\times n} \,|\, n \geq 1\}$, and $\{K_{pn} \,|\, n \geq 1\}$ respectively. Regular sequences are also constructed using 'levelled' Cayley graphs, based on a finite group. Some examples are given.

Keywords: removable, Cayley, isomorphism, reconstruction, clique

1 Notation, K_p-removable sequences of graphs, main results

For $p \geq 1$ and $n \geq 1$ let K_p be the complete graph on p vertices, let nK_p be n disjoint copies of K_p, and let $K_{p\times n} = K_{\underbrace{n,\ldots,n}_{p}}$, be the complete p-partite graph on pn vertices. All graphs G are simple. In the graph G let $G[\{v_1,\ldots,v_m\}] = G[v_1,\ldots,v_m]$ denote the induced subgraph on vertices $\{v_1,\ldots,v_m\}$. Suppose that for some p vertices $\{v_1,\ldots,v_p\}$ in G we have $G[v_1,\ldots,v_p] = K_p$, *i.e.*, $G[v_1,\ldots,v_p]$ is an induced K_p, then $G - K_p$ denotes the subgraph obtained from G by deleting vertices v_i and their incident edges, for every $i = 1,\ldots,p$. If two graphs G and G' are isomorphic we write $G \cong G'$. We often say that two graphs are 'equal' (=) instead of 'isomorphic', and say 'K_p' instead of 'induced K_p'.

For the countable sequence of graphs $\{G_{pn} \,|\, n \geq 1\} = \{G_{p1}, G_{p2}, G_{p3}, \ldots\}$ we use the notation $\{G_{pn}\}$, each graph G_{pn} has pn vertices for a fixed $p \geq 1$.

We call a sequence $\{G_{pn}\}$ K_p-*removable* if it satisfies the following two properties:

A1 $G_{p1} \cong K_p$,

A2 $G_{pn} - K_p \cong G_{p(n-1)}$ for every $n \geq 2$ and every (induced) K_p in G_{pn}.

In this paper we investigate K_p-removable sequences. In Section 2 we give a general construction for such sequences. In Section 3 we specialize to sequences in which each graph is regular; we call these regular (K_p, λ)-removable sequences, λ is a fixed number, $0 \leq \lambda \leq p$, referring to the fact that G_{pn} is $(\lambda(n-1)+p-1)$-regular. We classify regular $(K_p, 0)$-, $(K_p, p-1)$-, and (K_p, p)-removable sequences as the sequences $\{nK_p\}$, $\{K_{p\times n}\}$, and $\{K_{pn}\}$ respectively, thus associating three well-known graphs on pn vertices. In Section 4 we construct regular (K_p, λ)-removable sequences starting from a finite group; the graphs in this sequence are similar in construction to Cayley graphs, we also count the number of K_p's in these graphs.

2 Construction of a K_p-removable sequence for every $p \geq 3$, examples

From **A1** and **A2** above we see that if $\{G_{pn}\}$ is K_p-removable then $G_{p1} = K_p$ and G_{p2} is the union of this K_p and a 'new' K_p, together with some edges between them. The graph G_{p3} is then formed from G_{p2} by adding another K_p and some suitable set of edges between this new K_p and the previous two K_p's, and so on. Thus G_{pn} contains at least n disjoint K_p's. We use this idea of constructing G_{pn} by adding K_p to K_p, up to n K_p's, in the following:

Here $p \geq 3$ and $[p] = \{1, \ldots, p\}$. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_p)$ be an ordered p-tuple of subsets of $[p]$, *i.e.*, each $\Lambda_i \subseteq [p]$.

Consider a K_p with vertices labelled $\{(1,1), \ldots, (p,1)\} = \{(i,1) \,|\, i \in [p]\}$; call these vertices *vertices at level* 1, and call this graph H^{Λ}_{p1}. Now consider another K_p with vertices labelled $\{(i,2) \,|\, i \in [p]\}$, vertices at level 2. For any vertex $(i,2)$ join it to vertices $\{(i',1) \,|\, i' \in \Lambda_i\}$ at level 1; call this graph H^{Λ}_{p2}. Now consider a third K_p with vertices labelled $\{(i,3) \,|\, i \in [p]\}$, at level 3. Join any vertex $(i,3)$ to vertices $\{(i',2) \,|\, i' \in \Lambda_i\}$ at level 2 and to vertices $\{(i',1) \,|\, i' \in \Lambda_i\}$ at level 1; this is H^{Λ}_{p3}.

For any $n \geq 1$, consider the graph which has been constructed level by level, up to n levels, according to the above definition; call this graph H^{Λ}_{pn}. In H^{Λ}_{pn} the vertices are of the form (i,j) for every $i \in [p]$ and every $1 \leq j \leq n$, (where j is their level); and the edges are of two types:

(i) *fixed-level* edges, say at level j

$$((i_1, j), (i_2, j)) \text{ is an edge for all } i_1, i_2 \in [p] \text{ where } i_1 \neq i_2; \text{ and}$$

(ii) *cross-level* edges, for $j > j'$

$$((i, j), (i', j')) \text{ is an edge if and only if } i' \in \Lambda_i.$$

For each $i \in [p]$ let $\lambda_i = |\Lambda_i|$ be the number of elements in Λ_i, and let μ_i be the number of sets in Λ which contain i. Call Λ *uniform* if:

$$i \notin \Lambda_i \text{ and } \lambda_i = \mu_i \text{ for each } i \in [p]. \tag{1}$$

From now on let our Λ be uniform. In Theorem 2.3 we show that if Λ is uniform then $\{H^{\Lambda}_{pn}\}$ is K_p-removable.

For any fixed $i \in [p]$, let $I_i = \{(i,1), \ldots, (i,n)\} = \{(i,j) \,|\, 1 \leq j \leq n\}$ be the set of vertices of H^{Λ}_{pn} in 'column i'. Then, because $i \notin \Lambda_i$, this is an independent set of vertices.

Now let W be a K_p in H^{Λ}_{pn}, then each of the p independent sets $I_1, \ldots, I_p$ contains exactly one vertex from W; let I_i contain vertex $(i, w_i) \in W$, a vertex at level w_i, for some $1 \leq w_i \leq n$. Thus $W = H^{\Lambda}_{pn}[(1, w_1), \ldots, (p, w_p)]{=}K_p$.

Lemma 2.1 *Let Λ be uniform. For any $K_p = W$ in H^{Λ}_{pn} the number of edges in $H^{\Lambda}_{pn} - W$ equals the number of edges in $H^{\Lambda}_{p(n-1)}$.*

Proof. Consider any vertex (i,j) in H^{Λ}_{pn}, here $1 \leq j \leq n$. It is adjacent to λ_i vertices at each of the $j-1$ levels lower than level j, *i.e.*, to $\lambda_i(j-1)$ such vertices, and to $p-1$ vertices at level j, and to $\mu_i(n-j)$ vertices at levels higher than level j. Thus, because $\lambda_i = \mu_i$, its degree is

$$\deg((i,j)) = \lambda_i(n-1) + p - 1. \tag{2}$$

So if (i,j) is in $W = K_p$, then its degree 'outside' W is $\lambda_i(n-1)$, which is independent of its level, j.

Now W contains exactly one vertex from each independent set I_i, so, when removing W, we remove $\sum_{i=1}^{p}(\lambda_i(n-1))$ edges 'outside' W, (and $\binom{p}{2}$ 'inside' W). But this equals the number of edges removed if we remove the K_p at level n (because $\deg((i,n))$ is also given by (2)), leaving the graph $H^{\Lambda}_{p(n-1)}$. Hence, the number of edges in $H^{\Lambda}_{pn} - W$ equals the number of edges in $H^{\Lambda}_{p(n-1)}$. ■

Lemma 2.2 *Let Λ be uniform. For any $p \geq 3$, $n \geq 2$, and any $K_p = W$ in H^{Λ}_{pn}, we have*

$$H^{\Lambda}_{pn} - W = H^{\Lambda}_{p(n-1)}.$$

Proof. Let the vertices of W be $\{(i, w_i) \,|\, 1 \leq i \leq p\}$. We construct a bijection ϕ between the vertices of $H^{\Lambda}_{pn} - W$ and the vertices of $H^{\Lambda}_{p(n-1)}$, and then show that ϕ is an isomorphism. Under ϕ, for a fixed $i \in [p]$, the vertices in the i-th independent set of $H^{\Lambda}_{pn} - W$, namely in the set $I_i \backslash \{(i, w_i)\}$, are mapped to the vertices in the i-th independent set of $H^{\Lambda}_{p(n-1)}$, namely to the set $\{(i,1), \ldots, (i, n-1)\}$, as follows:

$$\begin{array}{lcl} (i,n) & \to & (i,n-1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ (i,w_i+1) & \to & (i,w_i) \\ (i,w_i-1) & \to & (i,w_i-1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ (i,1) & \to & (i,1) \end{array}$$

i.e.,

$$\phi(i,j) = \begin{cases} (i,j-1), & \text{for } w_i < j \leq n; \\ (i,j), & \text{for } 1 \leq j < w_i. \end{cases}$$

Clearly ϕ is a bijection.

There are two types of edges in H^{Λ}_{pn}: fixed-level edges and cross-level edges. First we deal with fixed-level edges.

A typical fixed-level edge in H^{Λ}_{pn} is $((i_1,j),(i_2,j))$ for some $i_1, i_2 \in [p]$ where $i_1 \neq i_2$ and for some j with $1 \leq j \leq n$. Thus, a typical fixed-level edge in $H^{\Lambda}_{pn} - W$ is $((i_1,j),(i_2,j))$ where $i_1 \neq i_2$, and $j \neq w_{i_1}$ and $j \neq w_{i_2}$, (because the vertices (i_1, w_{i_1}) and (i_2, w_{i_2}) have been removed). Without loss of generality let $w_{i_1} \leq w_{i_2}$.

Now we check that ϕ maps two adjacent vertices at level j in $H^{\Lambda}_{pn} - W$ onto two adjacent vertices in $H^{\Lambda}_{p(n-1)}$. There are three cases to consider:
(a) $1 \leq j < w_{i_1} \leq w_{i_2} \leq n$. Then $\phi((i_1,j),(i_2,j)) = (\phi(i_1,j),\phi(i_2,j)) = ((i_1,j),(i_2,j))$, which is certainly a (fixed-level) edge in $H^{\Lambda}_{p(n-1)}$.
(b) $1 \leq w_{i_1} < j < w_{i_2} \leq n$. Then $\phi((i_1,j),(i_2,j)) = ((i_1,j-1),(i_2,j))$. Now $w_{i_1} < w_{i_2}$ and $((i_1,w_{i_1}),(i_2,w_{i_2}))$ is an edge in W, and so in H^{Λ}_{pn}, so $i_1 \in \Lambda_{i_2}$; and $1 \leq j-1 < j \leq n-1$ and so $((i_1,j-1),(i_2,j))$ is a (cross-level) edge in $H^{\Lambda}_{p(n-1)}$.
(c) $1 \leq w_{i_1} \leq w_{i_2} < j \leq n$. Then $\phi((i_1,j),(i_2,j)) = ((i_1,j-1),(i_2,j-1))$, which, again, is a fixed-level edge in $H^{\Lambda}_{p(n-1)}$.

Cross-level edges in H^{Λ}_{pn} are of the form $((i,j),(i',j'))$ where $j > j'$ and $i' \in \Lambda_i$. Thus cross-level edges in $H^{\Lambda}_{pn} - W$ are of the form $((i,j),(i',j'))$, where $j > j'$, $j \neq w_i$, $j' \neq w_{i'}$, and $i' \in \Lambda_i$.

Now we check that ϕ maps two adjacent vertices at levels j and j' in $H_{pn}^{\Lambda} - W$ onto two adjacent vertices in $H_{p(n-1)}^{\Lambda}$. There are four cases to consider:

(a) $1 \le j < w_i \le n$ and $1 \le j' < w_{i'} \le n$. Then $\phi((i,j),(i',j')) = ((i,j),(i',j'))$, a cross-level edge in $H_{p(n-1)}^{\Lambda}$.
(b) $1 \le j < w_i \le n$ and $1 \le w_{i'} < j' \le n$. Then $\phi((i,j),(i',j')) = ((i,j),(i',j'-1))$, again, a cross-level edge in $H_{p(n-1)}^{\Lambda}$.
(c) $1 \le w_i < j \le n$ and $1 \le j' < w_{i'} \le n$. Then $\phi((i,j),(i',j')) = ((i,j-1),(i',j'))$; here $j-1 \ge j'$. If $j-1 > j'$ then this is a cross-level edge in $H_{p(n-1)}^{\Lambda}$, or, if $j-1 = j'$, then this is a fixed-level edge in $H_{p(n-1)}^{\Lambda}$.
(d) $1 \le w_i < j \le n$ and $1 \le w_{i'} < j' \le n$. Then $\phi((i,j),(i',j')) = ((i,j-1),(i',j'-1))$, a cross-level edge in $H_{p(n-1)}^{\Lambda}$.

Thus ϕ moves edges in $H_{pn}^{\Lambda} - W$ to edges in $H_{p(n-1)}^{\Lambda}$.

Now, from Lemma 2.1, the graphs $H_{pn}^{\Lambda} - W$ and $H_{p(n-1)}^{\Lambda}$ have the same number of edges, and so ϕ is an isomorphism. ∎

Thus we have the following existence result for K_p-removable sequences:

Theorem 2.3 *Let Λ be uniform. For any $p \ge 3$ the sequence $\{H_{pn}^{\Lambda}\}$ is K_p-removable.*

Proof. By construction, for every $n \ge 1$, the graph H_{pn}^{Λ} has pn vertices. Clearly the sequence $\{H_{pn}^{\Lambda}\}$ satisfies **A1**, and, from Lemma 2.2, it satisfies **A2**, hence it is K_p-removable. ∎

Example 1 $p = 3$, $\Lambda_1 = \{2\}$, $\Lambda_2 = \{1, 3\}$, $\Lambda_3 = \{2\}$. Here $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ is uniform with $\lambda_1 = \mu_1 = 1$, $\lambda_2 = \mu_2 = 2$, and $\lambda_3 = \mu_3 = 1$. The first 3 graphs in the K_3-removable sequence $\{H_{3n}^{\Lambda}\}$ are shown in Fig. 1.

The converse of Theorem 2.3 is not true, consider the example:

Example 2 $p = 3$, $\Lambda_1 = \{2\}$, $\Lambda_2 = \Lambda_3 = \emptyset$. Here $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$, and it is straightforward to show that $\{H_{3n}^{\Lambda}\}$ is K_3-removable but $\lambda_1 = 1$ and $\mu_1 = 0$, and so $\lambda_1 \ne \mu_1$ and Λ is not uniform.

We call a K_p-removable sequence $\{G_{pn}\}$ *regular* if all graphs in the sequence are regular, and *irregular* if at least one graph in the sequence is irregular.

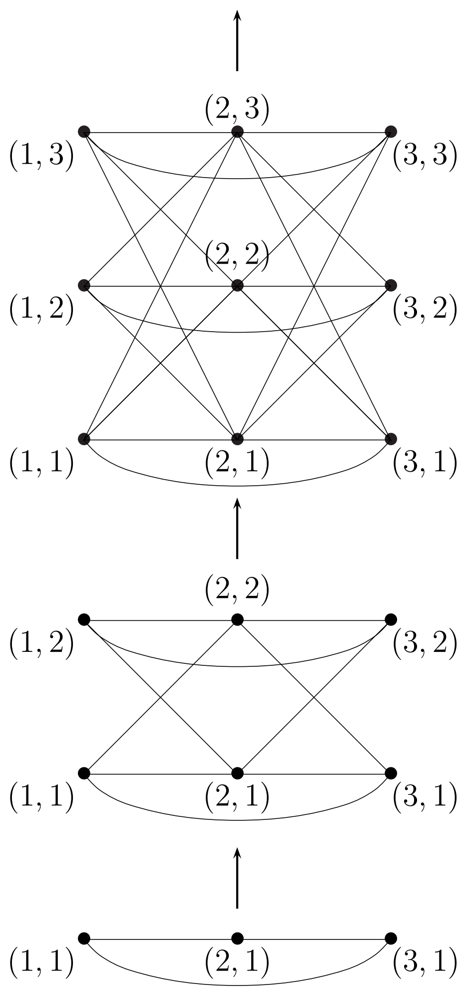


Figure 1: The first 3 graphs in the irregular K_3-removable sequence $\{H_{3n}^{\Lambda}\}$ where $\Lambda = \{\Lambda_1, \Lambda_2, \Lambda_3\}$ with $\Lambda_1 = \{2\}$, $\Lambda_2 = \{1, 3\}$, and $\Lambda_3 = \{2\}$. See Examples 1 and 3.

In Section 3 we show that all K_p-removable sequences for $p = 1$ and $p = 2$ are regular. As the next example shows, an irregular K_p-removable sequence exists for every $p \geq 3$.

Example 3 For a fixed $p \geq 3$, $\Lambda_1 = \{2\}$, $\Lambda_2 = \{1, 3\}$, $\Lambda_3 = \{2\}$, and $\Lambda_i = \emptyset$ for $4 \leq i \leq p$. Then $\Lambda = (\Lambda_1, \ldots, \Lambda_p)$ is uniform and so $\{H^{\Lambda}_{pn}\}$ is K_p-removable. Moreover, $\{H^{\Lambda}_{pn}\}$ is irregular because H^{Λ}_{p2} is irregular: $\deg((1, 2)) = p$ but $\deg((2, 2)) = p+1$. See Fig. 1 where $p = 3$, $\deg((1, 2)) = 3$ but $\deg((2, 2)) = 4$.

We are interested in the K_p's in H^{Λ}_{pn}. The next theorem gives necessary and sufficient conditions for their existence.

Let $V = H^{\Lambda}_{pn}[(1, v_1), \ldots, (p, v_p)]$ be an arbitrary induced subgraph in H^{Λ}_{pn} with exactly one vertex from each independent set I_i. Let V have vertices at m different levels: $\ell_1, \ldots, \ell_m$ where $\ell_1 < \cdots < \ell_m$. For $1 \leq k \leq m$, let $V_k = \{i \,|\, v_i = \ell_k\} \neq \emptyset$ be the set of first coordinates of all vertices of V at level ℓ_k . Then the sets $V_1, \ldots, V_m$ partition $[p] = \{1, \ldots, p\}$, and:

Theorem 2.4 *With the above notation $V = K_p$ if and only if for every k, with $1 \leq k \leq m$, we have*

$$V_1 \cup \cdots \cup V_k = \cap_{i \in V_{k+1} \cup \cdots \cup V_m} \Lambda_i,$$

where we define $\cap_{i \in \emptyset} \Lambda_i = [p]$.

Proof. Suppose that $V = K_p$, and suppose $i' \in V_1 \cup \cdots \cup V_k$ for some k with $1 \leq k \leq m$. Now vertex $(i', v_{i'})$ is at a lower level than all vertices (i, v_i) where $i \in V_{k+1} \cup \cdots \cup V_m$, so $i' \in \Lambda_i$ for all $i \in V_{k+1} \cup \cdots \cup V_m$. Thus $i' \in \cap_{i \in V_{k+1} \cup \cdots \cup V_m} \Lambda_i$, and $V_1 \cup \cdots \cup V_k \subseteq \cap_{i \in V_{k+1} \cup \cdots \cup V_m} \Lambda_i$. But, because, $i \notin \Lambda_i$, we have $\cap_{i \in V_{k+1} \cup \cdots \cup V_m} \Lambda_i \subseteq [p] \setminus \{V_{k+1} \cup \cdots \cup V_m\} = V_1 \cup \cdots \cup V_k$. Hence $V_1 \cup \cdots \cup V_k = \cap_{i \in V_{j+1} \cup \cdots \cup V_m} \Lambda_i$, as required. The converse is straightforward. ■

Example 4 See Example 1 and Fig. 1. Here $p = 3$, $\Lambda_1 = \{2\}$, $\Lambda_2 = \{1, 3\}$, and $\Lambda_3 = \{2\}$. Consider the induced graph $H^{\Lambda}_{33}[(1, 1), (2, 3), (3, 1)] = K_3$. This K_3 has vertices at $m = 2$ levels, *i.e.*, at level $\ell_1 = 1$ and at level $\ell_2 = 3$, so $V_1 = \{1, 3\}$ and $V_2 = \{2\}$. For $k = 1$ we have

$$V_1 = \cap_{i \in V_2} \Lambda_i = \Lambda_2,$$

and for $k = 2$

$$V_1 \cup V_2 = \cap_{i \in \emptyset} \Lambda_i = [3] = \{1, 2, 3\}.$$

Such a K_p with vertices at more than one level is a *cross-level* K_p.

3 Regular sequences, uniqueness of (K_p, λ)-removable sequences for $\lambda = 0$, $p - 1$, or p

Here we consider regular K_p-removable sequences $\{G_{pn}\}$, *i.e.*, those which also satisfy:

A3 G_{pn} is regular for every $n \geq 1$.

For such a sequence we use the notation $\{R_{pn}\}$, for every $n \geq 1$ R_{pn} is regular of degree $\deg(R_{pn})$. We know that $R_{p1} = K_p$, and the second graph in this sequence is R_{p2}, so let us define λ by

$$\lambda = \deg(R_{p2}) - \deg(R_{p1}) = \deg(R_{p2}) - (p - 1). \tag{3}$$

Lemma 3.1 *For a fixed $p \geq 1$ let $\{R_{pn}\}$ be K_p-removable with λ defined as above. Then*
(i) $0 \leq \lambda \leq p$,
(ii) $deg(R_{pn}) = \lambda(n - 1) + p - 1$ *for every* $n \geq 1$,
(iii) $\lambda = deg(R_{pn}) - deg(R_{p(n-1)})$ *for every* $n \geq 2$.

Proof. (i) By **A1** and **A2** the graph R_{p2} contains a K_p with $R_{p2} - K_p = R_{p1} = K_p$. By **A3** R_{p2} is regular and $\deg(R_{p2}) \geq \deg(K_p) = p - 1$, and so $\lambda = \deg(R_{p2}) - (p - 1) \geq 0$. Also, $\deg(R_{p2}) \leq \deg(R_{p1}) + p$ because the degree of a vertex in R_{p1} can be increased by at most p when constructing R_{p2} (if it is made adjacent to each of the p vertices in the new K_p), hence $\lambda \leq p$. Thus $0 \leq \lambda \leq p$, *i.e.*, $\lambda = 0, \ldots, p$.
(ii) Here we use induction on n. For $n = 1$ (ii) is true by **A1**, and for $n = 2$ it is true by (3). So assume that $n \geq 3$ and that (ii) is true for $n - 1$, *i.e.*, for the graph $R_{p(n-1)}$. Now R_{pn} is obtained from $R_{p(n-1)}$ by adding on a new K_p and some cross-level edges between them. But R_{pn} is regular, so we can count these cross-level edges in two different ways:

$$(n - 1)p\,[\deg(R_{pn}) - \deg(R_{p(n-1)})] = p\,[\deg(R_{pn}) - (p - 1)].$$

This gives

$$\begin{aligned}\deg(R_{pn}) &= \frac{(n-1)\deg(R_{p(n-1)}) - (p-1)}{n-2} \\ &= \lambda(n-1) + p - 1,\end{aligned}$$

using the induction hypothesis. Thus the induction goes through and (ii) is true for every $n \geq 1$.
(iii) This follows directly from (ii). ■

For a fixed $p \geq 1$ and a fixed λ where $\lambda = 0, \ldots, p$, let $\{R_{pn}\}$ be a K_p-removable sequence in which $\deg(R_{pn}) = \lambda(n-1)+p-1$ for every $n \geq 1$. We call the sequence $\{R_{pn}\}$ *regular* (K_p, λ)*-removable*, and denote it by $\{R^\lambda_{pn}\}$. Lemma 3.1(iii) then says that

$$\lambda = \deg(R^\lambda_{pn}) - \deg(R^\lambda_{p(n-1)}) \quad \text{for every } n \geq 2, \tag{4}$$

i.e., as we move from $R^\lambda_{p(n-1)}$ to R^λ_{pn} in the sequence $\{R^\lambda_{pn}\}$, the degree of regularity always increases by λ.

Now we look at some small values of p.

Example 5 $p = 1$, so $\deg(R^\lambda_{1n}) = \lambda(n-1)$ where $\lambda = 0$ or 1.

$\lambda = 0$ Here $\deg(R^0_{1n}) = 0$. So R^0_{1n} is the graph with n isolated vertices, nK_1. Clearly $R^0_{11} = K_1$ and, for $n \geq 2$, the removal of any K_1 from nK_1 gives the graph $(n-1)K_1$, thus $\{R^0_{1n}\} = \{nK_1\}$ is K_1-removable. It is also clear that $\{nK_1\}$ is the unique regular $(K_1, 0)$-removable sequence.

$\lambda = 1$ Here $\deg(R^1_{1n}) = n - 1$. $\{R^1_{1n}\} = \{K_n\}$ is the unique regular $(K_1, 1)$-removable sequence.

Example 6 $p = 2$, so $\deg(R^\lambda_{2n}) = \lambda(n-1) + 1$ where $\lambda = 0$, 1, or 2.

$\lambda = 0$ Here $\deg(R^0_{2n}) = 1$. $\{R^0_{2n}\} = \{nK_2\}$ is the unique regular $(K_2, 0)$-removable sequence.

$\lambda = 1$ Here $\deg(R^1_{2n}) = n$. Now $K_{n,n}$, the complete bipartite graph on $2n$ vertices, has degree n. Also $K_{1,1} = K_2$ and, for every $n \geq 2$, the removal of any K_2 from $K_{n,n}$ results in $K_{n-1,n-1}$, *i.e.* $K_{n,n} - K_2 = K_{n-1,n-1}$. So one example of a regular $(K_2, 1)$-removable sequence is $\{R^1_{2n}\} = \{K_{n,n}\}$, we show in Corollary 3.2 that this is the unique example.

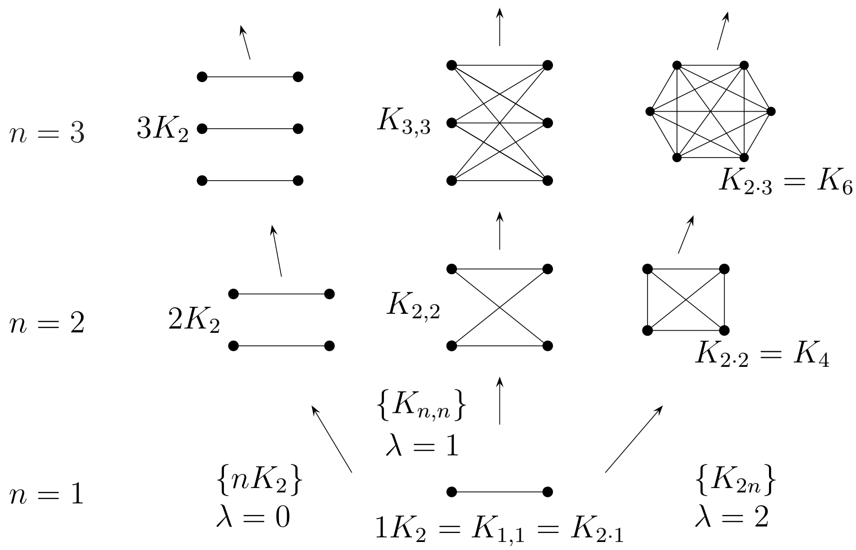


Figure 2: The first 3 graphs in each of the K_2-removable sequences $\{nK_2\}$, $\{K_{n,n}\}$, and $\{K_{2n}\}$. These are the unique regular $(K_2, 0)$-, $(K_2, 1)$-, and $(K_2, 2)$-removable sequences, respectively. See Example 6.

$\lambda = 2$ Here $\deg(R^2_{2n}) = 2n-1$. $\{R^2_{2n}\} = \{K_{2n}\}$ is the unique regular $(K_2, 2)$-removable sequence.

We see that the 1-st graph in each of the three sequences $\{R^0_{2n}\} = \{nK_2\}$, $\{R^1_{2n}\} = \{K_{n,n}\}$, and $\{R^2_{2n}\} = \{K_{2n}\}$ is K_2, *i.e.* we must think of K_2 as being the three graphs $1K_2 = K_{1,1} = K_{2\cdot 1}$. Then the 2-nd graph in each sequence is obtained by changing the 1's in the notation for these graphs into 2's, etc. This is illustrated in Fig. 2, where the first 3 graphs in each of the three sequences $\{nK_2\}$, $\{K_{n,n}\}$, and $\{K_{2n}\}$ are shown. Note also that the n-th graphs in each of the three sequences are well-known graphs on $2n$ vertices: nK_2, $K_{n,n}$, and K_{2n}, respectively.

Recall that a K_p-removable sequence $\{G_{pn}\}$ is regular if all of its graphs are regular, and is irregular if at least one of its graphs is irregular. The next result shows that all K_p-removable sequences for $p = 1$ or $p = 2$ are regular (and are those given in Examples 5 and 6). From Example 3 we see that irregular K_p-removable sequences exist for every $p \geq 3$.

Corollary 3.2 *For $p = 1$ or $p = 2$ all K_p-removable sequences are regular, (and are given in Examples 5 and 6).*

Proof. ($p = 2$) Let $\{G_{2n}\}$ be a K_2-removable sequence, then $G_{21} = K_2$. A candidate for G_{22} must be a graph on 4 vertices with at least two disjoint K_2's, and the removal of any K_2 must leave a K_2. The only possibilities are: (a) $2K_2$, (b) $K_{2,2}$, or (c) $K_{2\cdot 2} = K_4$, all of which are regular.

(a) We prove by induction on n that if $\{G_{2n}\}$ is an arbitrary K_2-removable sequence which begins $\{K_2, 2K_2, \ldots\}$ then it is the regular sequence $\{nK_2\}$.

Let $n \geq 3$ and suppose that $G_{2(n-1)} = (n-1)K_2$. Now consider G_{2n} which is the union of $(n-1)K_2$ and a 'new' K_2 (and some edges between them); let edge (u, v) be this new K_2. If (u, v) is an isolated K_2 then we are finished, so, assume that u is adjacent to vertex u_1 in $G_{2(n-1)}$. Now $G_{2(n-1)} = (n-1)K_2$ and $n \geq 3$ so $G_{2(n-1)}$ contains some edge (u_2, u_3) with $u_2 \neq u_1$ and $u_3 \neq u_1$. But $G_{2n} - (u_2, u_3)$ contains the path on the three vertices u_1, u, and v (or triangle if (v, u_1) is also an edge); a contradiction because $G_{2n} - (u_2, u_3) = (n-1)K_2$. Thus (u, v) must be an isolated K_2 and the induction goes through, *i.e.*, $G_{2n} = nK_2$ and so $\{G_{2n}\} = \{nK_2\}$.

(b) Similarly we use induction to show that any K_2-removable sequence $\{G_{2n}\}$ which begins $\{K_2 = K_{1,1}, K_{2,2}, \ldots\}$ is the regular sequence $\{K_{n,n}\}$.

Let $n \geq 3$ and suppose that $G_{2(n-1)} = K_{n-1,n-1}$; let the 2 independent sets of $G_{2(n-1)}$ be $I_1 = \{u_1, \ldots, u_{n-1}\}$ and $I_2 = \{v_1, \ldots, v_{n-1}\}$, and let the new K_2 of G_{2n} be the edge (u, v). Now (u, v) cannot be isolated so let vertex u be adjacent to some vertices in $G_{2(n-1)}$. Suppose that u is adjacent to vertex $u_{j_1} \in I_1$ and to vertex $v_{j_2} \in I_2$, then $G_{2n}[u, u_{j_1}, v_{j_2}] = C_3$, an odd cycle. Now remove any $K_2 = (u_{j_3}, v_{j_4})$ where $j_3 \neq j_1$ and $j_4 \neq j_2$, then $G_{2n} - (u_{j_3}, v_{j_4}) = K_{n-1,n-1}$ still contains this odd cycle, a contradiction. Thus u is adjacent to vertices from exactly one of I_1 or I_2, say I_2.

Now suppose that u is not adjacent to every vertex in I_2, suppose that it is not adjacent to v_{j_5}, say. Consider the edge (u_1, v_{j_6}) where $j_6 \neq j_5$. In the graph $G_{2n} - (u_1, v_{j_6}) = K_{n-1,n-1}$ the set of vertices $\{u_2, \ldots, u_{n-1}, u\} = \{I_1 \cup \{u\}\} \backslash \{u_1\}$ is one of the independent sets and $\{I_2 \cup \{v\}\} \backslash \{v_{j_6}\}$ is the other. Thus (u, v_{j_5}) is an edge, a contradiction. So u is adjacent to all vertices in I_2. Similarly v is adjacent to all vertices in I_1. Thus $G_{2n} = K_{n,n}$ and $\{G_{2n}\} = \{K_{n,n}\}$.

The proofs for (c) $K_{2\cdot 2} = K_4$ and the case $p = 1$ are similar. ∎

Before the next main result, Theorem 3.4, we need:

Lemma 3.3 *For a fixed* $p \geq 1$ *let* $\{R^\lambda_{pn}\}$ *be regular* (K_p, λ)*-removable. Then, for every* $n \geq 1$*,*
(i) R^λ_{pn} *contains* $\geq n$ *disjoint* K_p*'s,*
(ii) R^λ_{pn} *contains no* K_{p+1}*'s for* $0 \leq \lambda \leq p-1$*.*

Proof. (i) See the first paragraph of Section 2.
(ii) Let $0 \leq \lambda \leq p-1$. Suppose that for some $p+1$ vertices $\{v_1, \ldots, v_{p+1}\}$ of R^λ_{pn} we have $R^\lambda_{pn}[v_1, \ldots, v_{p+1}] = K_{p+1}$. Consider $R^\lambda_{pn} - K_p$ where $K_p = R^\lambda_{pn}[v_1, \ldots, v_p]$. Clearly $v_{p+1} \in R^\lambda_{pn} - K_p$ and $\deg(v_{p+1})$ has decreased by p; but from (4) it should have decreased by λ, a contradiction because $0 \leq \lambda \leq p-1$. ∎

Recall that $K_{p\times n} = K_{\underbrace{n,\ldots,n}_{p}}$ denotes the complete p-partite graph on pn vertices.

Theorem 3.4 *For a fixed* $p \geq 2$ *there is a unique regular* (K_p, λ)*-removable sequence for* $\lambda = 0$*,* $p-1$*, or* p*:*
(i) $\{nK_p\}$ *is the unique regular* $(K_p, 0)$*-removable sequence,*
(ii) $\{K_{p\times n}\}$ *is the unique regular* $(K_p, p-1)$*-removable sequence,*
(iii) $\{K_{pn}\}$ *is the unique regular* (K_p, p)*-removable sequence.*

Proof. The proofs of (i) and (iii) are straightforward.
(ii) ($\lambda = p-1$) We use induction on n. Let $\{R^{p-1}_{pn}\}$ be an arbitrary regular $(K_p, p-1)$-removable sequence; for $n = 1$ we know that $R^{p-1}_{p1} = K_p = K_{p\times 1}$. Let $n \geq 2$, then R^{p-1}_{pn} contains a $K_p = R^{p-1}_{pn}[v_1, \ldots, v_p]$, say. By the induction hypothesis we have $R^{p-1}_{pn} - K_p = R^{p-1}_{p(n-1)} = K_{p\times(n-1)}$. Let $I_1, \ldots, I_p$ be the p independent sets of $R^{p-1}_{p(n-1)}$, each has $n-1$ vertices. Now suppose that in R^{p-1}_{pn} a vertex from $\{v_1, \ldots, v_p\}$, say v_i, is adjacent to a vertex from each of these p independent sets; call these vertices $\{u_1, \ldots, u_p\}$, where $u_k \in I_k$ for $k = 1, \ldots, p$. Then $R^{p-1}_{pn}[v_i, u_1, \ldots, u_p] = K_{p+1}$, a contradiction to Lemma 3.3(ii). Thus v_i is adjacent to vertices in at most $p-1$ of the independent sets $I_1, \ldots, I_p$.

Now v_i is adjacent to $(p-1)(n-1)$ vertices from $I_1 \cup \cdots \cup I_p$, so it must be adjacent to all $n-1$ vertices in some $p-1$ of $I_1, \ldots, I_p$. Suppose that in R^{p-1}_{pn} two distinct vertices from $\{v_1, \ldots, v_p\}$, say v_{i_1} and v_{i_2}, are adjacent to all of the vertices in the same $p-1$ independent sets, say $I_1, \ldots, I_{p-1}$. Let $u_k \in I_k$ for $k = 1, \ldots, p-1$. Then $R^{p-1}_{pn}[v_{i_1}, v_{i_2}, u_1, \ldots, u_{p-1}] = K_{p+1}$, again a contradiction. Thus, any two distinct vertices from $\{v_1, \dots, v_p\}$ are adjacent to all vertices in distinct $(p-1)$-sets of $\{I_1, \dots, I_p\}$.

For $i = 1, \dots, p$ let v_i be non-adjacent to all vertices in the independent set $I_{i'}$ for some $i' = 1, \dots, p$. By above the mapping $v_i \leftrightarrow I_{i'}$ is a bijection. Then $\{v_i\} \cup I_{i'}$ is an independent set in R_{pn}^{p-1}, and v_i is adjacent to all the vertices in all other independent sets. This is true for every $i = 1, \dots, p$. It is now clear that $R_{pn}^{p-1} = K_{p\times n}$, and so the induction goes through and $\{R_{pn}^{p-1}\} = \{K_{p\times n}\}$. ∎

Note that this theorem connects three well-known graphs on pn vertices: nK_p, $K_{p\times n}$, and K_{pn}. The case $p = 2$ was discussed in Example 6 and illustrated in Fig. 2. Note also that the three regular sequences (i) $\{nK_p\}$, (ii) $\{K_{p\times n}\}$, and (iii) $\{K_{pn}\}$ are the sequences $\{H_{pn}^{\Lambda}\}$ where (i) $\Lambda = (\underbrace{\emptyset, \dots, \emptyset}_{p})$, (ii) $\Lambda = ([p]\backslash\{1\}, \dots, [p]\backslash\{p\})$, and (iii) $\Lambda = ([p], \dots, [p])$, respectively. (In (iii) Λ is not uniform.)

So, for $p = 1$ or $p = 2$ and all values of λ, and for every $p \geq 3$ and $\lambda = 0$, $p-1$, or p, there is a unique regular (K_p, λ)-removable sequence.

4 Levelled Cayley Graphs, more regular (K_p, λ)-removable sequences

Generally our construction of $\{H_{pn}^{\Lambda}\}$ from Section 2 gives irregular sequences. For an arbitrary vertex $(i, j) \in H_{pn}^{\Lambda}$ we have $\deg((i, j)) = \lambda_i(n-1) + p - 1$, see (2). So H_{pn}^{Λ} is regular if and only if λ_i is constant for all $i \in [p]$. We set $\lambda_i = \lambda$ then H_{pn}^{Λ} is regular of degree $\lambda(n-1) + p - 1$, (see Lemma 3.1(ii)), and $\{H_{pn}^{\Lambda}\}$ is a regular (K_p, λ)-removable sequence.

We now construct a regular (K_p, λ)-removable sequence using a finite group. This combines the construction of $\{H_{pn}^{\Lambda}\}$ from Section 2 with the construction of a Cayley graph; see, for example, p. 122 of Biggs [2].

Let $p \geq 3$ and let $\mathcal{G}_p = \{g_1, \dots, g_p\}$ be a finite group with p elements, where $g_1 = e$ is the identity element. Let $\Lambda \subseteq \mathcal{G}_p$ be a subset of $\mathcal{G}_p$ with $e \notin \Lambda$ and $|\Lambda| = \lambda$.

Now consider the following graph, a *levelled Cayley* graph, $\Gamma_n = \Gamma_n(\mathcal{G}_p, \Lambda)$: It has $n \geq 1$ levels of vertices, each level having p vertices. For any fixed j

with $1 \le j \le n$ the vertices at level j are $\{(g_1, j), \ldots, (g_p, j)\} = \{(g, j) \mid g \in \mathcal{G}_p\}$, and the edges are of two types:

(i) *fixed-level* edges, say at level j

$$((g, j), (h, j)) \text{ is an edge for all } g, h \in \mathcal{G}_p \text{ where } g \neq h;$$

($i.e.$, the 'fixed-level' graph is K_p), and
(ii) *cross-level* edges, for $j > j'$

$$((g, j), (g', j')) \text{ is an edge if and only if } g'g^{-1} \in \Lambda.$$

Using Theorem 2.3 it is straightforward to prove:

Theorem 4.1 *For any finite group $\mathcal{G}_p$ with p elements and any $\Lambda \subseteq \mathcal{G}_p$ with $e \notin \Lambda$ and $|\Lambda| = \lambda$, the sequence $\{\Gamma_n(\mathcal{G}_p, \Lambda)\}$ is regular (K_p, λ)-removable.*

■

Now, for any $\mathcal{G}_p$ with $p \ge 3$ and for each $\lambda = 0, \ldots, p-1$, there exists a Λ satisfying the requirements in Theorem 4.1, and Theorem 3.4(iii) takes care of $\lambda = p$, so we have the following existence result for regular (K_p, λ)-removable sequences.

Theorem 4.2 *For any $p \ge 3$ and any $\lambda = 0, \ldots, p$, there exists a regular (K_p, λ)-removable sequence, namely the sequence $\{\Gamma_n(\mathcal{G}_p, \Lambda)\}$, where $\mathcal{G}_p$ is any finite group of order p and Λ is any subset of $\mathcal{G}_p$ with $e \notin \Lambda$ and $|\Lambda| = \lambda$.*

■

Let $\overline{\Lambda}$ denote the complement of Λ in $\mathcal{G}_p$ and let $\langle\overline{\Lambda}\rangle$ be the subgroup generated by $\overline{\Lambda}$, also let $\langle\overline{\Lambda}\rangle g$ denote a typical coset of this subgroup.

Let $V = \Gamma_n[(g_1, v_1), \ldots, (g_p, v_p)]$ be an arbitrary induced subgraph in $\Gamma_n(\mathcal{G}_p, \Lambda)$ with exactly one vertex from each independent set $I_i = \{(g_i, j) \mid 1 \le j \le n\}$, where $i = 1, \ldots, p$. We are interested in the K_p's in $\Gamma_n(\mathcal{G}_p, \Lambda)$. The next theorem gives a necessary and sufficient condition for this V to be a K_p, this condition is cleaner than the corresponding condition of Theorem 2.4 for the graph H^{Λ}_{pn}; it also enables us to count the number of K_p's in $\Gamma_n(\mathcal{G}_p, \Lambda)$.

As in Section 2 let V have vertices at m different levels: $\ell_1 < \cdots < \ell_m$. For $1 \le k \le m$, let $V_k = \{g_i \mid v_i = \ell_k\} \neq \emptyset$ be the set of first coordinates of all vertices of V at level ℓ_k. Then the sets $V_1, \ldots, V_m$ partition $\mathcal{G}_p$, and we have:

Theorem 4.3 *With the above notation $V = K_p$ if and only if for every k, with $1 \leq k \leq m$, V_k is a union of cosets of $\langle\overline{\Lambda}\rangle$.*

Proof. Suppose that $V = \Gamma_n[(g_1, v_1), \ldots, (g_p, v_p)] = K_p$. Now $\ell_m = \max\{v_1, \ldots, v_p\}$ is the highest level of $\Gamma_n(\mathcal{G}_p, \Lambda)$ which contains vertices from V, and $V_m = \{g_i \,|\, v_i = \ell_m\}$ is the set of first coordinates of all vertices of V at this highest level.

Let s be an arbitrary element in $V_m \subseteq \mathcal{G}_p$. Now consider the graph $V' = \Gamma_n[(g_1 s^{-1}, v_1), \ldots, (g_p s^{-1}, v_p)]$, let us reorder the vertices in V' so that $V' = \Gamma_n[(g_1, v'_1), \ldots, (g_p, v'_p)]$; we use this version of V'. The graph V' is also a K_p and the highest level of its vertices is ℓ_m also. Then $V'_m = \{g_i \,|\, v'_i = \ell_m\} = V_m s^{-1}$ is the set of first coordinates of these vertices. We show that V'_m is a union of cosets of $\langle\overline{\Lambda}\rangle$.

First we show that $\overline{\Lambda} \subseteq V'_m$. Now $g_1 = e \in \overline{\Lambda}$ and $s \in V_m$, so certainly $e = ss^{-1} \in V'_m$. So vertex $(g_1, v'_1) = (e, \ell_m) \in V'$ lies at level ℓ_m. Let $g_i \neq e$ and let $g_i \in \overline{\Lambda}$, but suppose that $g_i \notin V'_m$. This means that vertex $(g_i, v'_i) \in V'$ does not lie at level ℓ_m so it must lie at a lower level, *i.e.*, $\ell_m > v'_i$. Now, because $((e, \ell_m), (g_i, v'_i))$ is an edge in V' then $((s, \ell_m), (g_i s, v'_i))$ is an edge in V and so in $\Gamma_n(\mathcal{G}_p, \Lambda)$; so $g_i s s^{-1} = g_i \in \Lambda$, a contradiction because $g_i \in \overline{\Lambda}$. Thus for every $g_i \in \overline{\Lambda}$, then $g_i \in V'_m$, *i.e.*, $\overline{\Lambda} \subseteq V'_m$.

Next we show that $\langle\overline{\Lambda}\rangle \subseteq V'_m$. For any $r \geq 1$ let $\prod(r)$ denote a product of r arbitrary elements from $\overline{\Lambda}$. By induction we show, for any fixed $r \geq 1$, that every $\prod(r) \in V'_m$. Now $\overline{\Lambda} \subseteq V'_m$, *i.e.*, every $\prod(1) \in V'_m$. Assume for some r with $r \geq 1$ that every $\prod(r) \in V'_m$. Now suppose that some $\prod(r+1)$, say $h(r+1) = a_1 \cdots a_{r+1} \notin V'_m$, but each $a_1, \ldots, a_{r+1} \in \overline{\Lambda}$. Let $h(r) = a_2 \cdots a_{r+1}$, then, by the induction hypothesis, $h(r) \in V'_m$ so vertex $(h(r), \ell_m) \in V'$ is at the highest level ℓ_m and the vertex in V' with first coordinate $h(r+1)$ is at a lower level. There is an edge between these two vertices so $h(r+1)h(r)^{-1} = a_1 \in \Lambda$, a contradiction because $a_1 \in \overline{\Lambda}$. Thus $h(r+1) \in V'_m$ and the induction goes through, and so $\langle\overline{\Lambda}\rangle \subseteq V'_m$.

Finally we show that V'_m is a union of cosets of $\langle\overline{\Lambda}\rangle$. Let $g \in V'_m$, but suppose that $g \notin \langle\overline{\Lambda}\rangle$. Then, by similar reasoning to before, we see that every $\prod(1)g \in V'_m$, and by induction, that every $\prod(r)g \in V'_m$ for every $r \geq 1$; thus the coset $\langle\overline{\Lambda}\rangle g \subseteq V'_m$. Thus V'_m is a union of cosets of $\langle\overline{\Lambda}\rangle$, and so $V_m = V'_m s$ is also.

Now we return to the graph V and show that V_k is a union of cosets of $\langle\overline{\Lambda}\rangle$ for every $1 \leq k \leq m$. From above this is true for V_m, so assume for all k with $k \leq m$ that V_k is a union of cosets of $\langle\overline{\Lambda}\rangle$, we show that V_{k-1} is also.

Let $g \in V_{k-1}$, we show that the coset $\langle\overline{\Lambda}\rangle g \subseteq V_{k-1}$. As before first we show that every $\prod(1)g \in V_{k-1}$, for, suppose not, then:

Either $\prod(1)g \in V_{k'}$ where $k' > k-1$, so, by the induction hypothesis, $V_{k'}$ is a union of cosets of $\langle\overline{\Lambda}\rangle$. Thus $\langle\overline{\Lambda}\rangle \prod(1)g = \langle\overline{\Lambda}\rangle g \subseteq V_{k'}$, so $g \in V_{k'}$, a contradiction because $g \in V_{k-1}$;

Or $\prod(1)g \in V_{k'}$ where $k' < k-1$, *i.e.*, level $\ell_{k'}$ is lower than level ℓ_{k-1}. So vertex $(\prod(1)g, \ell_{k'}) \in V$ is at a lower level than vertex $(g, \ell_{k-1}) \in V$, so $\prod(1)gg^{-1} = \prod(1) \in \Lambda$, again a contradiction.

So $\prod(1)g \in V_{k-1}$, and we proceed by induction as before to show that for any $r \geq 1$ then every $\prod(r)g \in V_{k-1}$ and so $\langle\overline{\Lambda}\rangle g \subseteq V_{k-1}$, and the induction on k goes through; thus V_{k-1} is a union of cosets of $\langle\overline{\Lambda}\rangle$.

So, in conclusion, for any k with $1 \leq k \leq m$, V_k is a union of cosets of $\langle\overline{\Lambda}\rangle$, as required.

For the converse let each V_k be a union of cosets of $\langle\overline{\Lambda}\rangle$. Let (g, ℓ_k) and $(g', \ell_{k'})$ be two arbitrary vertices in V, we show that $((g, \ell_k), (g', \ell_{k'}))$ is an edge in $\Gamma_n(\mathcal{G}_p, \Lambda)$. If $\ell_k = \ell_{k'}$ then, certainly, $((g, \ell_k), (g', \ell_{k'}))$ is an edge by construction of $\Gamma_n(\mathcal{G}_p, \Lambda)$. Otherwise, without loss of generality, let $\ell_k > \ell_{k'}$. Then g and g' are in different cosets of $\langle\overline{\Lambda}\rangle$, so $g'g^{-1} \notin \langle\overline{\Lambda}\rangle$, so $g'g^{-1} \in \overline{\langle\overline{\Lambda}\rangle} \subseteq \Lambda$, and again $((g, \ell_k), (g', \ell_{k'}))$ is an edge. Thus $V = K_p$ as required. ∎

Now we count the number of K_p's in $\Gamma_n(\mathcal{G}_p, \Lambda)$; let $|\mathcal{G}_p : \langle\overline{\Lambda}\rangle|$ be the index of $\langle\overline{\Lambda}\rangle$ in $\mathcal{G}_p$, *i.e.*, the number of cosets of $\langle\overline{\Lambda}\rangle$ in $\mathcal{G}_p$.

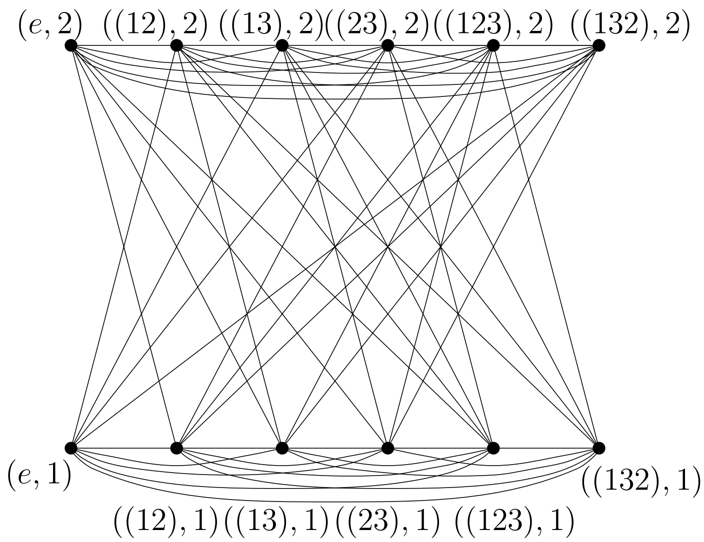
Corollary 4.4 *The number of K_p's in $\Gamma_n(\mathcal{G}_p, \Lambda)$ is given by*

$$n^{|\mathcal{G}_p : \langle\overline{\Lambda}\rangle|}.$$

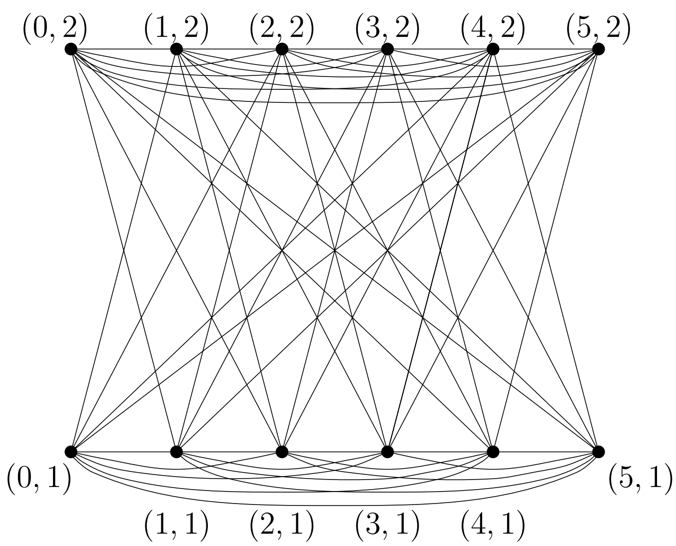
Proof. Consider any coset $\langle\overline{\Lambda}\rangle g$, let us 'place' the elements of this coset at any fixed level j, where $1 \leq j \leq n$, in the graph $\Gamma_n(\mathcal{G}_p, \Lambda)$. Each such placement of every coset of $\langle\overline{\Lambda}\rangle$ gives a K_p and every K_p corresponds to such a placement of every coset of $\langle\overline{\Lambda}\rangle$. Hence, the number of K_p's in $\Gamma_n(\mathcal{G}_p, \Lambda)$ equals the number of such placements of all the cosets of $\langle\overline{\Lambda}\rangle$. There are $|\mathcal{G}_p : \langle\overline{\Lambda}\rangle|$ cosets, and n levels to place each, hence $n^{|\mathcal{G}_p : \langle\overline{\Lambda}\rangle|}$ such placements and so $n^{|\mathcal{G}_p : \langle\overline{\Lambda}\rangle|}$ corresponding K_p's. ∎

Example 7 (a) Let $\mathrm{Sym}(3) = \{e, (12), (13), (23), (123), (132)\}$ be the symmetric group on $\{1, 2, 3\}$, and let $\Lambda = \{(12), (13), (23), (123)\}$. Then $\overline{\Lambda} = \{e, (132)\}$ and $\langle\overline{\Lambda}\rangle = \{e, (123), (132)\}$; the other coset of $\langle\overline{\Lambda}\rangle$ is $\langle\overline{\Lambda}\rangle(12) = \{(12), (13), (23)\}$. Consider the graph $\Gamma_2(\mathrm{Sym}(3), \Lambda)$ shown in Fig. 3(a), we have $|\mathrm{Sym}(3) : \langle\overline{\Lambda}\rangle| = 2$ so this graph has $2^2 = 4$ K_6's. They are the 2 fixed-level K_6's, $\Gamma_2[(e, 1), ((123), 1), ((132), 1), ((12), 2), ((13), 2), ((23), 2)]$, and $\Gamma_2[(e, 2), ((123), 2), ((132), 2), ((12), 1), ((13), 1), ((23), 1)]$. This is the 2-nd graph in the regular $(K_6, 4)$-removable sequence $\{\Gamma_n(\mathrm{Sym}(3), \Lambda)\}$.
(b) Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be the additive group (mod 6), and let $\Lambda = \{1, 2, 4, 5\}$. Then $\Lambda = \langle\overline{\Lambda}\rangle = \{0, 3\}$ and the other cosets of $\langle\overline{\Lambda}\rangle$ are $\{1, 4\}$ and $\{2, 5\}$. Thus $|\mathbb{Z}_6 : \langle\overline{\Lambda}\rangle| = 3$ and $\Gamma_2(\mathbb{Z}_6, \Lambda)$ has $2^3 = 8$ K_6's, see Fig. 3(b). One such K_6 is $\Gamma_2[(0, 2), (3, 2), (1, 1), (4, 1), (2, 1), (5, 1)]$, and another is $\Gamma_2[(0, 2), (3, 2), (1, 1), (4, 1), (2, 2), (5, 2)]$, in which the vertices corresponding to coset $\{2, 5\}$ have been moved up one level. This is the 2-nd graph in the regular $(K_6, 4)$-removable sequence $\{\Gamma_n(\mathbb{Z}_6, \Lambda)\}$.

Note that the two graphs in Example 7 are non-isomorphic because they have a different number of K_6's, they are the 2-nd graphs in two different regular $(K_6, 4)$-removable sequences. So, in general, regular $(K_p, p-2)$-removable sequences are not unique.



(a)



(b)

Figure 3: The non-isomorphic graphs (a) $\Gamma_2(\mathrm{Sym}(3), \Lambda)$ where $\Lambda = \{(12), (13), (23), (123)\}$, and (b) $\Gamma_2(\mathbb{Z}_6, \Lambda)$ where $\Lambda = \{1, 2, 4, 5\}$. See Example 7.

Example 8 Consider the graphs $\Gamma_3(\mathbb{Z}_4, \{1\})$ and $\Gamma_3(\mathbb{Z}_4, \{2\})$, shown in Figs. 4(a) and (b) respectively. Both graphs have 3 K_4's. In $\Gamma_3(\mathbb{Z}_4, \{1\})$ the union of all edges which lie outside its 3 K_4's is a C_{12}, however in $\Gamma_3(\mathbb{Z}_4, \{2\})$ this union is $C_6 \cup C_6$. Thus $\Gamma_3(\mathbb{Z}_4, \{1\}) \neq \Gamma_3(\mathbb{Z}_4, \{2\})$ and so $\{\Gamma_n(\mathbb{Z}_4, \{1\})\} \neq \{\Gamma_n(\mathbb{Z}_4, \{2\})\}$, and we have two different regular $(K_4, 1)$-removable sequences.

Example 8 illustrates that, in general, regular $(K_p, 1)$-removable sequences are not unique.

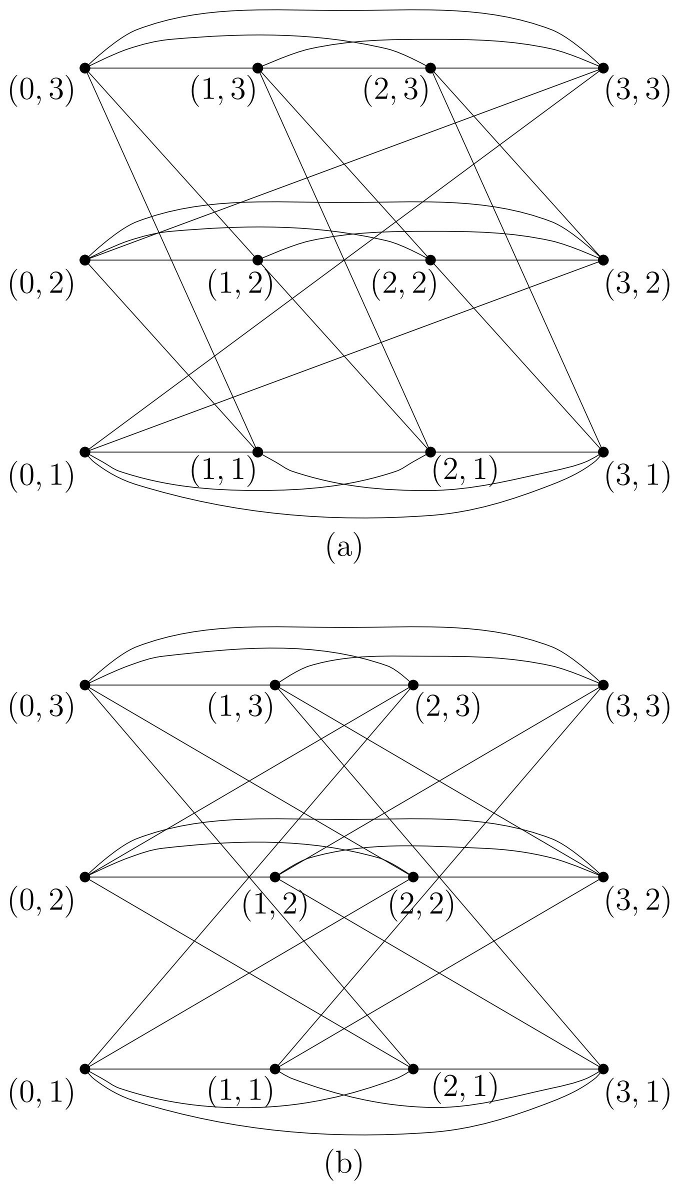


Figure 4: The non-isomorphic graphs (a) $\Gamma_3(\mathbb{Z}_4, \{1\})$ and (b) $\Gamma_3(\mathbb{Z}_4, \{2\})$. See Example 8.

Theorem 3.4 states that, for a fixed $p \geq 2$, the only regular $(K_p, 0)$-, $(K_p, p-1)$-, and (K_p, p)-removable sequences are unique. Above we give two different examples of a regular $(K_p, 1)$-removable sequence, and two different examples of a regular $(K_p, p-2)$-removable sequence; thus, in general, regular (K_p, λ)-removable sequences are not unique unless $\lambda = 0$, $p-1$, or p.

Some final comments:

If, for some $\mathcal{G}_p$ and Λ we have $|\mathcal{G}_p : \langle\overline{\Lambda}\rangle| = 1$, then the graph $\Gamma_n(\mathcal{G}_p, \Lambda)$ has exactly n K_p's which is the least number allowed by Lemma 3.3(i). Such a graph has no cross-level K_p's; *e.g.*, $\Gamma_3(\mathbb{Z}_4, \{1\})$ shown in Fig. 4(a).

It is also worth mentioning that for any group $\mathcal{G}_p$: (see Theorem 3.4)
(i) $\{\Gamma_n(\mathcal{G}_p, \emptyset)\} = \{nK_p\}$ is the unique regular $(K_p, 0)$-removable sequence,
(ii) $\{\Gamma_n(\mathcal{G}_p, \mathcal{G}_p \backslash \{e\})\} = \{K_{p\times n}\}$ is the unique regular $(K_p, p-1)$-removable sequence,
(iii) $\{\Gamma_n(\mathcal{G}_p, \mathcal{G}_p)\} = \{K_{pn}\}$ is the unique regular (K_p, p)-removable sequence. (Note that in (iii) we do not have $e \notin \Lambda$.) Thus the unique regular (K_p, λ)-removable sequences for $\lambda = 0$, $p-1$, or p can all be constructed from an arbitrary group $\mathcal{G}_p$.

For other papers on graph sequences see Barefoot, Entringer, and Jackson [1], and the references therein; another somewhat related paper is Duchet, Tuza, and Vestergaard [3].

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