Clique-Symmetric Uniform Hypergraphs

John P. McSorley
Department of CCTM,
London Metropolitan University,
100 Minories, London. EC3N 1JY
mcsorley60@hotmail.com

Thomas D. Porter
Department of Mathematics,
Southern Illinois University,
Carbondale. IL 62901-4408
tporter@math.siu.edu

Abstract

Let H be an r-uniform hypergraph of order p, and $\{H_{p1}, H_{p2}, \ldots\}$ be a countable sequence of r-uniform hypergraphs with H_{pn} having pn vertices. The sequence is H-removable if $H_{p1} \cong H$ and $H_{pn} - S \cong H_{p(n-1)}$ where S is any vertex subset of H_{pn} that induces a copy of H. This paper deals with the case $H = K_p^r$. It provides a construction of hypergraphs with a high degree of symmetry; where for any such hypergraph, all the ways of removing the vertices of any fixed number of disjoint K_p^r's yields the same subgraph. The case $r = 2$ was studied by the authors in [3]. This paper gives the generalization to r-uniform hypergraphs for all $r = 2, 3, \ldots$.

Keywords: clique, eulerian digraph, hypergraph, m-partite, Stirling number

1 The Case $\mathbf{r = 2}$

We first briefly list the previous ideas in [3] and [4] concerning the case $r = 2$. The same vocabulary and definitions will be needed in our subsequent sections.

In general we follow the notation in [5]. In particular if W is a subset of the vertices of a graph G, then $G[W]$ denotes the subgraph of G induced by W. The digraphs and graphs we consider are loopless and without multiple arcs or edges. We also use $[p] = \{1, \ldots, p\}$.

Call a countable sequence of graphs $\{G_{pn}\} = \{G_{p1}, G_{p2}, \ldots\}$ *K_p-removable* if it satisfies the following two properties:

P1: $G_{p1} \cong K_p$

P2: $G_{pn} - W \cong G_{p(n-1)}$ for every $n \geq 2$ and every vertex subset $W \subset V(G_{pn})$ that induces a K_p in G_{pn}, *i.e.*, $G_{pn} \cong K_p$.

We often write $G_1 = G_2$ in place of $G_1 \cong G_2$.

Let $\vec{D}$ be a digraph of order p, with $d^+(u) = d^-(u)$ for every vertex u in $V(\vec{D})$. Then $\vec{D}$ is an eulerian digraph if $\vec{D}$'s underlying 'undirected' graph is of one component, otherwise $\vec{D}$ is eulerian on each of its underlying components. Let $N^+(i)$ denote the out-neighborhood of vertex i.

Consider a copy of K_p with vertices labelled $\{(1,1), \ldots, (p,1)\} = \{(i,1) \,|\, i \in [p]\}$; call these vertices *vertices at level* 1, and call this graph $D_1(K_p)$. Now consider another copy of K_p with vertices labelled $\{(i,2) \,|\, i \in [p]\}$, these are vertices at level 2. For any vertex $(i,2)$ join it to the vertices $\{(i',1) \,|\, i' \in N^+(i)\}$ at level 1, so we see that these edges are derived from the digraph $\vec{D}$. We call the graph so formed $D_2(K_p)$. Now consider a third K_p with vertices labelled $\{(i,3) \,|\, i \in [p]\}$, at level 3. Join any vertex $(i,3)$ to vertices $\{(i',2) \,|\, i' \in N^+(i)\}$ at level 2 and to vertices $\{(i',1) \,|\, i' \in N^+(i)\}$ at level 1; this is $D_3(K_p)$.

Now, for any $n \geq 1$, consider the graph which has been constructed level by level, up to n levels, according to the previous definition; call this graph $D_n(K_p)$ or simply D_n when p is clear. We say the digraph $\vec{D}$ *generates* the sequence $\{D_n\}$. The vertices of D_n, $V(D_n)$, are of the form (i,j) for every $i \in [p]$ and every $1 \leq j \leq n$, where j is their level; and the edges are of two types:

(i) *fixed-level* edges, say at level j

$$((i_1, j), (i_2, j)) \text{ is an edge for all } i_1, i_2 \in [p] \text{ where } i_1 \neq i_2; \text{ and}$$

(ii) *cross-level* edges, for $j > j'$

$$((i,j),(i',j')) \text{ is an edge if and only if } i' \in N^+(i).$$

For any fixed $i \in [p]$, let $I_i = \{(i,1),\dots,(i,n)\} = \{(i,j) \mid 1 \le j \le n\}$ be the set of vertices of D_n in 'column i'. Then, because $i \notin N^+(i)$, *i.e.*, because $\vec{D}$ doesn't have loops, this is an independent set of vertices. Now let W be a subset of $V(D_n)$ that induces a p-clique; then each of the p independent sets $I_1,\dots,I_p$ contain exactly one vertex from W.

Let $W = \{(1,v_1),\dots,(p,v_p)\}$ be an arbitrary vertex subset in D_n with exactly one vertex from each independent set I_i. Let W have vertices at m different levels: $\ell_1,\dots,\ell_m$ where $\ell_1 < \cdots < \ell_m$. For $1 \le k \le m$, let $V_k = \{i \mid v_i = \ell_k\} \ne \emptyset$ be the set of first coordinates of all vertices of W at level ℓ_k. Then the sets $V_1,\dots,V_m$ partition $[p]$.

In D_n consider two levels of vertices, V_i and V_j with $l_i < l_j$ and let $y \in V_i$ and $x \in V_j$. Then if the edge $e = xy$ is in the induced subgraph $D_n[W]$ of D_n we call the arc (x,y) in our original generating digraph $\vec{D}$ a W-*skew arc*. Hence a W-skew arc of $\vec{D}$ gives rise to edges in D_n which join different levels of W.

Let (A,B) denote the set of arcs in D from A to B, *i.e.*, all arcs (a,b) with $a \in A$ and $b \in B$.

Theorem 1.1 ([4]). *With the above notation: a set W of vertices of D_n with level-partition $V_1, V_2, \dots, V_m$, induces a p-clique iff the associated W-skew arcs form a complete symmetric m-partite subdigraph in $\vec{D}$.* □

Theorem 1.2 ([4]). *Let $\vec{D}$ be any eulerian digraph of order p. Then its generated sequence of graphs $\{D_n\}$ is K_p-removable.* □

2 The Hypergraph Construction

The ideas and results in Section 1 lead to the following extension to hypergraphs. A hypergraph consists of a collection of vertices and a collection of edges; if the vertex set is V, then the edges are subsets of V. A hypergraph is r-*uniform* if all of its edges have size r. The complete r-uniform hypergraph of order p, denoted by K_p^r, is the hypergraph with vertex set $V = [p]$ and with edges all of the $\binom{p}{r}$ r-subsets of V.

For a fixed pair p and r, with $p \geq r \geq 2$, let $\{H_{pn}^r\} = \{H_{p1}^r, H_{p2}^r, H_{p3}^r, \ldots\}$ be a sequence of r-uniform hypergraphs where H_{pn}^r has pn vertices. Such a sequence is called K_p^r-*removable* if it satisfies the following properties:

P1 $H_{p1}^r \cong K_p^r$

P2 $H_{pn}^r - K_p^r \cong H_{p(n-1)}^r$ for every $n \geq 2$ and for every (induced) K_p^r in H_{pn}^r.

For each pair p and r, with $p \geq r \geq 2$, we show the existence of K_p^r-removable sequences.

Let $(A_1, \ldots, A_m)$ denote a partition of $[p]$. We define the complete m-partite r-uniform hypergraph of order p, denoted $K_{|A_1|,\ldots,|A_m|}^r$, as follows: The vertex set is $[p]$, and an r-subset, Q, of $[p]$ is an edge if and only if Q contains at most $r-1$ members from any class A_j. So, *eg.*, $K_{n,n}^r$ has $2n$ vertices and

$$\binom{n}{1}\binom{n}{r-1} + \binom{n}{2}\binom{n}{r-2} + \cdots + \binom{n}{r-1}\binom{n}{1} = \binom{2n}{r} - 2\binom{n}{r}$$

edges. Notice that for $r = 2$ we have the usual $K_{n,n}^2$, the complete bipartite graph with n^2 edges. The number of edges in $K_{|A_1|,\ldots,|A_m|}^r$ is

$$e(K_{|A_1|,\ldots,|A_m|}^r) = \sum \binom{|A_1|}{k_1} \cdots \binom{|A_m|}{k_m}$$

where the sum is over all $k_1 + \cdots + k_m = r$, with $0 \leq k_j < r$ for all $1 \leq j \leq m$.

Motivated by Theorem 1.1 we now construct K_p^r-removable sequences, $\{H_{pn}^r\}$, from each fixed partition $(A_1, \ldots, A_m)$ of $[p]$.

With $\{H^r_{pn}\} = \{H^r_{p1}, H^r_{p2}, \ldots\}$ the construction is as follows:

1. $H^r_{p(m-1)} \cong (m-1)K^r_p$, *i.e.*, we first start with $m-1$ disjoint levelled copies of K^r_p.

2. For $n \geq m$, the graph H^r_{pn} is defined as follows. First, take n disjoint levelled copies of K^r_p; the notation for the vertices introduced at level 1, level 2, ..., level n, is the same as in Section 1, *eg.*, the copy of K^r_p at level j has vertex set $\{(1,j), (2,j), \ldots, (p,j)\}$. From the given partition $(A_1, \ldots, A_m)$ of $[p]$, to each A_j, $1 \leq j \leq m$, we select a distinct level l_{A_j}, where $1 \leq l_{A_j} \leq n$. We use $(A_1, \ldots, A_m) \to (l_{A_1}, \ldots, l_{A_m})$ to denote the selected levels. Notice that to the partition $(A_1, \ldots, A_m)$ there are $m!\binom{n}{m}$ such level $(l_{A_1}, \ldots, l_{A_m})$ selections.

   To each *fixed* level selection $(A_1, \ldots, A_m) \to (l_{A_1}, \ldots, l_{A_m})$ we identify the set A_j with their corresponding vertices in level l_{A_j}. The identification is through the first coordinates of the vertices in l_{A_j}. For example, if $A_j = \{x_1, \ldots, x_{|A_j|}\}$, then we identify A_j with the vertices, $\tilde{A}_j = \{(x_1, l_{A_j}), (x_2, l_{A_j}), \ldots, (x_{|A_j|}, l_{A_j})\}$ in level l_{A_j} of H^r_{pn}. We then have $(A_1, \ldots, A_m) \cong (\tilde{A}_1, \ldots, \tilde{A}_m)$ with $|\tilde{A}_1| + \cdots + |\tilde{A}_m| = p$, for each of the $m!\binom{n}{m}$ such $(\tilde{A}_1, \ldots, \tilde{A}_m)$'s. We then add to the initial n disjoint copies of K^r_p, all edges in $\bigcup E(K^r_{|\tilde{A}_1|, \ldots, |\tilde{A}_m|})$, where the union is over all $m!\binom{n}{m}$ such $(\tilde{A}_1, \ldots, \tilde{A}_m)$'s. We call this graph H^r_{pn}, the hypergraph *generated* by the partition $(A_1, \ldots, A_m)$.

We use the symbol $\text{gen}(A_1, \ldots, A_m)^r$ to denote such a sequence $\{H^r_{pn}\}$. Notice by the construction, from $(A_1, \ldots, A_m)$ and a chosen level-$(\tilde{A}_1, \ldots, \tilde{A}_m)$, the vertices $\tilde{A}_1 \cup \tilde{A}_2 \ldots \cup \tilde{A}_m$ in $V(H^r_{pn})$ induce a K^r_p. The other K^r_p's in H^r_{pn} are 'fixed-level' cliques, *i.e.*, the K^r_p introduced at each level $1, \ldots, n$. For the vertex subsets I_i, as defined earlier, the construction yields edges which contain at most one vertex from any 'column i'. Hence I_i is an independent set in H^r_{pn}. Also notice that each induced K^r_p contains exactly one vertex from each I_i, $1 \leq i \leq p$.

For $r = 2$, the connection to the digraph $\vec{D}$ in Theorem 1.2 is as follows: Consider any complete m-partite graph $G = K^2_{|A_1|, \ldots, |A_m|}$. Let $\vec{D}$ be the digraph formed from G by replacing each edge xy in G with two arcs (x, y) and (y, x). Then $\vec{D}$ is eulerian and it generates the K^2_p-removable sequence $\{D_n\}$ as in Theorem 1.2. Suppose W induces a K^r_p in H^r_{pn}. Let the vertices of W be $\{(i, w_i) \,|\, 1 \leq i \leq p\}$. In the graph $H^r_{pn} - W$, the set $I_i \backslash \{(i, w_i)\}$ is an independent set: call this the i-th independent set of $H^r_{pn} - W$. Now we construct an isomorphism ϕ between the vertices of $H^r_{pn} - W$ and the vertices of $H^r_{p(n-1)}$. Under ϕ, for a fixed $i \in [p]$, the vertices in the i-th independent set of $H^r_{pn} - W$, namely in the set $I_i \backslash \{(i, w_i)\}$, are mapped to the vertices in the i-th independent set of $H^r_{p(n-1)}$, namely to the set $\{(i,1), \ldots, (i, n-1)\}$, as follows:

$$\phi(i,j) = \begin{cases} (i, j-1), & \text{for } w_i < j \leq n \\ (i,j), & \text{for } 1 \leq j < w_i. \end{cases}$$

In [3] it is shown that ϕ is an isomorphism. It is straightforward to show that ϕ moves edges in $H^r_{pn} - W$ to edges in $H^r_{p(n-1)}$. We have:

Theorem 2.1. *Let $p \geq r \geq 2$, and let $(A_1, \ldots, A_m)$ be any partition of $[p]$. Then the sequence of hypergraphs gen$(A_1, \ldots, A_m)^r$ is a K^r_p-removable sequence.* □

In the following let $S(m,t)$ be the *Stirling numbers of the second kind.*

Notice, by the above construction of $\text{gen}(A_1, \ldots, A_m)^r = \{H^r_{pn}\}$, H^r_{pn} contains essentially two types of induced K^r_p's, either *fixed-level* or *cross-level.* The vertices of a K^r_p are either *all* at a fixed level, or are at *exactly* m different levels. However, by the isomorphism ϕ, the removal of any one of these *two* types of induced K^r_p's yields up to isomorphism the same sub-hypergraph. So, the hypergraph construction gives the clique-symmetric uniform hypergraph we were searching for that does not simply possess n fixed-level type K^r_p's.

We remark that the $\text{gen}(A_1, \ldots, A_m)^r$ construction can be slightly modified to also contain cross-level K^r_p's containing vertices from *exactly* t levels, for all $2 \leq t \leq m-1$, as follows: The partition $(A_1, \ldots, A_m)$ of $[p]$ is given. For fixed t, $2 \leq t \leq m-1$, let $(b_1, \ldots, b_t)$ be a partition of $[m]$. From $(b_1, \ldots, b_t)$ define the corresponding $(B_1, \ldots, B_t)$ where $B_j = \bigcup_{k \in b_j} A_k$. Notice that $(B_1, \ldots, B_t)$ is then a partition of $[p]$. Also notice that $K^r_{|B_1|, \ldots, |B_t|}$ is a complete t-partite sub-hypergraph of $K^r_{|A_1|, \ldots, |A_m|}$; and there are $t!S(m,t)$ such possible $(b_1, \ldots, b_t)$'s, and hence $t!S(m,t)$ such possible $(B_1, \ldots, B_t)$'s. We remark that in general, for any complete m-partite graph, where $m \geq 3$, there are $S(m,t)$ complete t-partite spanning subgraphs, where $2 \leq t \leq m-1$. As before, each $(B_1, \ldots, B_t)$ is assigned a level-selection $(\ell_{B_1}, \ldots, \ell_{B_t})$ producing the corresponding $(\tilde{B}_1, \ldots, \tilde{B}_t)$. Now define $X_t = \bigcup E(K^r_{|\tilde{B}_1|, \ldots, |\tilde{B}_t|})$, where the union is over all $(\tilde{B}_1, \ldots, \tilde{B}_t)$'s. For the *generalized* construction from a given $(A_1, \dots, A_m)$, we again start with n disjoint levelled K_p^r's. We add to these the edges $X_2 \cup X_3 \cdots \cup X_m$. It can be shown the isomorphism ϕ works for this graph as well.

We call such a K_p^r-removable sequence $\{H_{pn}^r\}$ as constructed above a t-gen$(A_1, \dots, A_m)^r$ sequence. We note that for a given $(B_1, \dots, B_t)$ and its associated $t!\binom{n}{t}$ corresponding $(\tilde{B}_1, \dots, \tilde{B}_t)$'s, that any permutation $\sigma(B_1, \dots, B_t)$ of $(B_1, \dots, B_t)$, yields the same set of $(\tilde{B}_1, \dots, \tilde{B}_t)$'s as $(B_1, \dots, B_t)$ does. Hence there are $t!S(m,t)\binom{n}{t}$ such $(\tilde{B}_1, \dots, \tilde{B}_t)$'s, for each $2 \le t \le m$. By the construction every K_p^r in H_{pn}^r is in one-to-one correspondence with a fixed $(\tilde{B}_1, \dots, \tilde{B}_t)$. For the case $t = 1$, we interpret the $(\tilde{B}_1)$'s as the beginning n-level copies of K_p^r, so $X_1 \cong nK_p^r$. Hence we have a rather nice formula for the number of K_p^r's in H_{pn}^r.

Theorem 2.2. *Let $(A_1, \dots, A_m)$ be a partition of $[p]$. For the K_p^r-removable sequence t-gen$(A_1, \dots, A_m)^r = \{H_{pn}^r\}$, the hypergraph H_{pn}^r has $\sum_{t=1}^{m} t!S(m,t)\binom{n}{t}$ $= n^m$ K_p^r's.* □

More directly, the value n^m in Theorem 2.2 can be interpreted as follows: From the given initial partition $(A_1, \dots, A_m)$ of $[p]$ we embed each A_j (by our definition of $\tilde{A}_j$) into any of the n levels, level 1, …, level n, thus creating $(\tilde{B}_1, \dots, \tilde{B}_t)$. Hence there are n^m such embeddings. We remark also that, by the defined construction, for each embedding $(\tilde{B}_1, \dots, \tilde{B}_t)$ the vertices $\tilde{B}_1 \cup \cdots \cup \tilde{B}_t$ induce a K_p^r in H_{pn}^r.

Example Let $p = 5$, $r = 3$, and (A_1, A_2, A_3) be the partition of $\{1,2,3,4,5\}$ with $A_1 = \{1,2\}$, $A_2 = \{3,5\}$, and $A_3 = \{4\}$. Consider the t-gen$(A_1, A_2, A_3)^3 = \{H_{5n}^3\}$ sequence. The graph for $n = 3$, *i.e.*, $H_{(5)(3)}^3$ has $1!S(3,1)\binom{3}{1}+2!S(3,2)\binom{3}{2}+3!S(3,3)\binom{3}{3} = 27$ induced K_5^3's. An example of one of these cliques for $t = 2$ is as follows: with $(B_1, B_2) = (A_1 \cup A_3, A_2)$, let $(B_1, B_2) \to (\ell_{B_1}, \ell_{B_2}) = (3,2)$ be a level-selection, giving $(\tilde{B}_1, \tilde{B}_2)$. Then the vertices in $V(H_{(5)(3)}^3)$, namely, $\tilde{B}_1 \cup \tilde{B}_2 = \{(3,1),(3,2),(3,4),(2,3),(2,5)\}$, induce a K_5^3 in $H_{(5)(3)}^3$.

K_{pn}^r is the complete r-uniform hypergraph on pn vertices.

We end by showing, with the exception of $\{K_{pn}^r\}$, that for any K_p^r-removable sequence $\{H_{pn}^r\}$, with $p \ge r \ge 2$, any member H_{pn}^r of the sequence does not contain a K_{p+1}^r. So, *e.g.*, for the case $r = 2$, the clique number is always $\omega(H_{pn}^2) = p$ for all n. We remark that the constructions given in this paper produce ways to generate K_p^r-removable sequences, we do not claim these are the *only* K_p^r-removable sequences. However, our final theorem applies to any K_p^r-removable sequence.

Theorem 2.3. *Suppose the n-th member H_{pn}^r of a K_p^r-removable sequence contains a K_{p+1}^r, then H_{pn}^r is K_{pn}^r.*

Proof. Suppose that H_{pn}^r contains a K_{p+1}^r. Without loss of generality, we assume $V(H_{pn}^r)$ is partitioned into n K_p^r's: $L_1, L_2, \ldots, L_n$, so that some vertex y in L_2 is such that $L_1 \cup \{y\}$ is an induced K_{p+1}^r in H_{pn}^r. Let x be any vertex in L_1. Deleting the $n-1$ K_p^r's: $L_3, L_4, \ldots, L_n$, $L_1 + \{y\} - \{x\}$, in this order, we obtain $L_2 + \{x\} - \{y\}$. Since $\{H_{pn}^r\}$ is K_p^r-removable, $L_2 + \{x\} - \{y\}$ is necessarily a K_p^r. Hence, $L_2 \cup \{x\}$ is a K_{p+1}^r, and the union of L_1 and L_2 is K_{2p}^r. Consequently the removal of any $n-2$ disjoint K_p^r's must produce K_{2p}^r. This implies that the union of every two levels L_i and L_j induce a K_{2p}^r; therefore, H_{pn}^r is K_{pn}^r. □

For other papers on graph sequences see Barefoot, Entringer, and Jackson [1], and its bibliography; see also Duchet, Tuza, and Vestergaard [2].

References

[1] C.A. Barefoot, R.C. Entringer, D.E. Jackson, *Graph theoretic modelling of cellular development II*, Proc. 19-th Southeastern Internat. Conf. on Combinatorics, Graph Theory, and Computing, (Baton Rouge, LA, 1988), Congr. Numer. 65 (1988) 135–146.

[2] P. Duchet, Z. Tuza, P.D. Vestergaard, *Graphs in which all spanning subgraphs with r fewer edges are isomorphic*, Proc. 19-th Southeastern Internat. Conf. on Combinatorics, Graph Theory, and Computing, (Baton Rouge, LA, 1988), Congr. Numer. 67 (1988) 45–57.

[3] J.P. McSorley and T.D. Porter, *K_p-removable sequences of graphs*, to appear, Journal of Combinatorial Mathematics and Combinatorial Computing.

[4] J.P. McSorley and T.D. Porter, *Generating sequences of clique-symmetric graphs via Eulerian digraphs*, to appear, Discrete Mathematics.

[5] D.B. West, Introduction to Graph Theory, 2nd Edition, (Prentice Hall, 2001).