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Math 305 **First Order Linear Equations**

A first order differential equation can be linear in either y or t. An ODE is linear in y if it can ultimately be written in the form

$$y' + p(t)y = f(t)$$

Steps: 1. Identify the coefficient of y'.

2. Is the derivative of this function equal to the coefficient of y? If yes, go to 7. If no, go to 3.

3. Divide by coefficient of y'.

4. Now identify $p(t)$ (the coefficient of y).

5. Evaluate $\int_t p(s)ds$.

6. Multiply the ODE through by $e^{\int_t p(s)ds}$.

7. The left-hand side can be written as the product of the coefficient of y' times y.
   Write it like this.

8. Integrate both sides.

9. Plug in IC.

Example A. $t^2y' + 2ty = \sin t$. $y\left(\frac{\pi}{2}\right) = 1$.

1. The coefficient of y' is t^2.

2. $\frac{d}{dt}t^2 = 2t =$coefficient of y. Go to 7.

7. $t^2y' + 2ty = \frac{d}{dt}(t^2y) = \sin t$

8. $\int d(t^2y) = \int \sin t dt$ so that $t^2y = -\cos t + c$.

9. Plug in IC: $\frac{\pi^2}{4} \cdot 1 = 0 + c$. Thus $t^2y = -\cos t + \frac{\pi^2}{4}$.

Example B. $(\sin t)y' + (\cos t)y = e^t \quad y\left(\frac{\pi}{4}\right) = 0$

1. The coefficient of y' is $\sin t$.

2. $\dfrac{d}{dt}\sin t = \cos t =$coefficient of y. Go to 7.

7. $(\sin t)y' + (\cos t)y = \dfrac{d}{dt}(y \sin t) = e^t$.

8. $\displaystyle\int d(y\sin t) = \int e^t dt$ so that $y\sin t = e^t + c$.

9. Plug in IC: $0\sin\dfrac{\pi}{4} = e^{\pi/4} + c$. Thus $y\sin t = e^t - e^{\pi/4}$.

Example C. $2y' + y = t \quad y(0) = 1$.

1. The coefficient of y' is 2.

2. $\dfrac{d}{dt}2 = 0 \neq$ coefficient of y.

3. Divide ODE by 2.

4. $p(t) = \dfrac{1}{2}$.

5. $\displaystyle\int \frac{1}{2} ds = \frac{1}{2}t$

6. $e^{t/2}y' + \dfrac{1}{2}e^{t/2}y = (e^{t/2})\dfrac{t}{2}$

7. $e^{t/2}y' + \dfrac{1}{2}e^{t/2}y = \dfrac{d}{dt}\left(e^{t/2}y\right) = (e^{t/2})\dfrac{t}{2}$

8. $\displaystyle\int d\left(e^{t/2}y\right) = \int \frac{t}{2}e^{t/2}dt$ so that

$$ye^{1/2t} = te^{1/2t} - 2e^{1/2t} + c$$

9. Plugging in IC $1 = 0 - 2 + c$. Thus $ye^{t/2} = te^{t/2} - 2e^{t/2} + 3$.

An equation is linear in t if it can be written as

$$t' + p(y)t = f(y).$$

The procedure is the same as the above replacing t by y and y by t.

Example D. $t' + (\tan y)t = \sin y$. $y(0) = 0$

1. The coefficient of t' is 1.
2. $\frac{d}{dy}(1) = 0 \neq \tan y =$ coefficient of t.
3. Divide ODE by 1.
4. $p(y) = \tan y$
5. $\int_y p(s)ds = -\ln\cos y = \ln\sec y$
6. Note $e^{\ln\sec y} = \sec y$. This is what we will multiply the ODE through by $(\sec y)t' + (\tan y \sec y)t = \sin y \sec y$.
7. $(\sec y)t' + (\tan y \sec y)t = \frac{d}{dy}((\sec y)t) = \sin y \sec y$
8. $\int d(t\sec y) = \int \frac{\sin y}{\cos y}dy$ so that $t\sec y = -\ln\cos y + c$.
9. Plugging in IC: $0\sec 0 = -\ln\cos 0 + c$. Thus $t\sec y = -\ln\cos y$.

Problems:

1. $e^{t^2}y' + 2te^{t^2}y = \tan t \qquad y(0) = 1$
2. $t^{-3}y' - \frac{3}{t^4}y = t \qquad y(1) = 3$
3. $(\sin y)t' + (\cos y)t = y^2 + 1 \qquad y(0) = \frac{\pi}{2}$
4. $y^2t' + 2yt = \sin y \qquad y(0) = \frac{\pi}{4}$
5. $(\cos t)y' + (\sin t)y = 1 \qquad y\left(\frac{\pi}{4}\right) = 2$
6. $t' + \frac{1}{y}t = \cos y \qquad y(2) = 1$
7. $\frac{1}{2}y' - ty = e^{t^2}\sin^2 t \qquad y(0) = 1$
8. $(\sin y)t' - (\cos y)t = \sin y \qquad y(1) = \frac{\pi}{4}$

Hint I: Note that when an equation is linear in y, and you have written the left-hand side as the derivative of a product, the right-hand side should only involve t! If not, something is wrong.

Hint II: By now you should realize that t, y, x, z, etc. are what we call "dummy" variables. You can have equations linear in any of these variables with one of the other variables the independent variable. A first order ODE is said to be linear if the ODE is linear in the dependent variable.

Examples:

9. $y^3 y' = \dfrac{y^4}{3t + y^5}$
10. $z' + (\cot s) z = 1$
11. $y^2 x' + 2yx = e^y$
12. $t' = \dfrac{t}{t^3 - w}$
13. $t' - \dfrac{1}{x} t = x^3 \sin x$

Solutions:

1. $y = e^{-t^2} \ln|\sec t| + e^{-t^2}$
2. $y = \frac{1}{2} t^5 + \frac{5}{2} t^3$
3. $t = \frac{1}{3} y^3 \csc y + y \csc y - \left(\dfrac{\pi}{2} + \dfrac{1}{3} \left(\dfrac{\pi}{2} \right)^3 \right) \csc y$
4. $t = -\dfrac{1}{y^2} \cos y + \dfrac{\sqrt{2}}{2y^2}$
5. $y = \sin t + (2\sqrt{2} - 1) \cos t$
6. $t = \sin y + \dfrac{1}{y} \cos y + (2 - \sin 1 - \cos 1) \dfrac{1}{y}$
7. $y = te^{t^2} - \dfrac{1}{2} e^{t^2} \sin 2t + e^{t^2}$
8. $t = -\sin y \ln|\csc y + \cot y| + (\sqrt{2} + \ln(\sqrt{2} + 1)) \sin y$
9. $t = \dfrac{1}{2} y^5 + c y^3$
10. $z = -\cos s + c \csc s$
11. $x = y^{-2} e^y + c y^{-2}$
12. $w = \dfrac{t^3}{4} + \dfrac{c}{t}$
13. $t = -x^3 \cos x + 2x^2 \sin x + 2x \cos x + cx$