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Math 305 **Bernoulli Equation**

A first order differential equation can be Bernoulli in either variable. A Bernoulli equation in y would be written in the form

$$y' + p(t)y = f(t)y^n.$$

A Bernoulli equation in t would be written in the form

$$t' + p(y)t = f(y)t^n.$$

We will look at the first case. The basic idea is to make a change of variables and reduce this nonlinear equation to a linear equation.

Steps:

1. Let $v(t) = (y(t))^{1-n}$.
2. Compute the derivative $\frac{dv}{dt} = (1-n)y^{-n}\frac{dy}{dt}$.
3. Solve for $\frac{dy}{dt} = \frac{y^n}{1-n}\frac{dv}{dt}$ and substitute into the ODE.
4. Divide by y^n.
5. Change y^{1-n} terms into v.
6. The equation is now linear in v.

Example A. $y' + \frac{1}{2t}y = \frac{\sin t}{2t}y^{-1}$

1. The $v(t) = (y(t))^{1-(-1)} = (y(t))^2$
2. $\frac{dv}{dt} = 2y\frac{dy}{dt}$.
3. $\frac{dy}{dt} = \frac{1}{2y}\frac{dv}{dt}$ so that $\frac{1}{2y}\frac{dv}{dt} + \frac{1}{2t}y = \frac{\sin t}{2ty}$
4. $\frac{1}{2}\frac{dv}{dt} + \frac{1}{2t}y^2 = \frac{\sin t}{2t}$
5. $\frac{1}{2}\frac{dv}{dt} + \frac{1}{2t}v = \frac{\sin t}{2t}$ Now solve as a linear equation. You still have work to do!

Example B. $(\sin y)t' - (\cos y)t = yt^2$

1. $v(y) = (t(y))^{1-2} = (t(y))^{-1}$
2. $\dfrac{dv}{dy} = -1t^{-2}\dfrac{dt}{dy}$.
3. $\dfrac{dt}{dy} = -t^2\dfrac{dv}{dy}$ so that $(\sin y)\left(-t^2\dfrac{dv}{dy}\right) - (\cos y)t = yt^2$
4. $(-\sin y)\dfrac{dv}{dy} - (\cos y)t^{-1} = y$
5. $-(\sin y)\dfrac{dv}{dy} - (\cos y)v = y$ Can you finish this?? Hint: It's linear in v!

Problems:

1. $y' + y = t^2e^ty^{1/2} \qquad y(0) = 4$
2. $y' - xy = \dfrac{1}{2}e^{-x^2}y^3 \qquad y(1) = 1$
3. $2x' - x = \dfrac{e^s}{sx} \qquad x(1) = 4$
4. $s^2t' + st = \dfrac{\ln s}{2t} \qquad t(e) = 3$
5. $(\sin x)y' + 2(\cos x)y = xy^{3/2} \qquad y\left(\dfrac{\pi}{4}\right) = 1$

Solutions:

1. $e^{1/2t}y^{1/2} = t^2e^{t/2} - 4te^{t/2} + 8e^{t/2} - 4$
2. $\dfrac{e^{x^2}}{y^2} = x + e - 1$
3. $x^2e^{-s} = \ln s - 16e^{-1}$
4. $s^2t^2 = s\ln s - s + 9e^2$
5. $\dfrac{y^{-1/2}}{\sin x} = x\cot x - \ln|\sin x| + \sqrt{2} - 1 + \ln\sqrt{2}$