Math 305 **Higher Dimensions: Functions of More than One Variable**

In Calculus I and II you studied functions of one independent variable. In Calculus III functions of more than one independent variable are discussed. For example, the function $f(x, y)$ has x and y as the independent variables and f (or z) as the dependent variable.

The same topics of limits and continuity are talked about. But being in three or more dimensions, the pictures and concepts are slightly more difficult. Of course the next topics that naturally come to mind are differentiation, then closely followed by integration. In this class we only need be concerned with functions of two independent variables so that is the context in which differentiation and integration will be discussed. Although going to 3 or 4 or more independent variables is not mind expanding.

Partial Derivatives

When taking derivatives of a function of more than one variable, the derivatives are called partial derivatives. The idea is to take the derivative with respect to one variable treating the other as a constant. For example, $\dfrac{\partial f}{\partial x}$ or f_x, means you take the derivative with respect to x treating y as a constant. Pictorially stand at a place on the y-axis and see how x changes. Computationally you ignore the y and literally treat it as a constant. $\dfrac{\partial f}{\partial y}$ or f_y means you take the derivative with respect to y treating x as a constant. Pictorially stand at a place on the x-axis and see how y changes. Computationally you ignore the x and literally treat it as a constant.

Example A. Consider $f(x, y) = x^2 + 3xy + y^2$

$$\frac{\partial f}{\partial x} = f_x = 2x + 3y$$
$$\frac{\partial f}{\partial y} = f_y = 3x + 2y$$

Example B. Consider $f(x, y) = xe^y + \tan^{-1} \dfrac{x}{y}$

$$\frac{\partial f}{\partial x} = f_x = e^y + \frac{1}{1 + \left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right)$$

$$\frac{\partial f}{\partial y} = f_y = xe^y + \frac{1}{1 + \left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)$$

Example C. Consider $f(x,y) = e^{xy}$

$$\begin{aligned} \frac{\partial f}{\partial x} &= f_x = ye^{xy} \\ \frac{\partial f}{\partial y} &= f_y = xe^{xy} \end{aligned}$$

Example D. Consider $f(x,y) = x^2y + 3xy^3$

$$\begin{aligned} \frac{\partial f}{\partial x} &= f_x = 2xy + 3y^3 \\ \frac{\partial f}{\partial y} &= f_y = x^2 + 9xy^2 \end{aligned}$$

Of course higher derivatives can be taken as well, i.e.,

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \end{aligned}$$

Problems: Find $f_x,\ f_y,\ f_{xx},\ f_{xy},\ f_{yx},\ f_{yy}$ if

1. $f(x,y) = \sin(xy)$
2. $f(x,y) = \ln x + y^3$
3. $f(x,y) = \tan^{-1}\dfrac{y}{x}$
4. $f(x,y) = x \ln y$
5. $f(x,y) = (x^2+y^2)^{1/2}$
6. $f(x,y) = e^{x^2} + \sin x - \cos y + \ln y$
7. $f(x,y) = \ln\dfrac{x}{y}$
8. $f(x,y) = xe^{xy} + y \sin xy$

Integration when integrand involves function of more than one variable

The basic idea with integration is similar to differentiation. If you're integrating with respect to x you treat y as a constant. If you're integrating with respect to y you treat x as a constant.

Hint I: When integrating with respect to x, the "constant" of integration is now no longer a constant, rather it is a function of y. Remember when you differentiate a function of y with respect to x you get zero.

Hint II: Remember to check your answer by differentiating!

Example A: $\int (2x+3y)dx = x^2 + 3xy + h(y)$.

Example B: $\int ye^{xy}dx = e^{xy} + h(y)$

Example C: $\int \sin x dy = y\sin x + h(x)$

Example D: $\int (x^2 + e^{xy})dy = x^2y + \frac{1}{x}e^{xy} + h(x)$

Problems: Evaluate:

9. $\int (x^2 + 3y + \sin x)dx$

10. $\int (3x + y - e^y)dy$

11. $\int (ye^{xy} + \tan x)dx$

12. $\int \left(xe^{xy} + \frac{1}{y}\right) dy$

13. $\int \left(\sin(x+y) + \frac{y}{x^2+y^2} + x\right) dx$

14. $\int \left(\sin(x+y) + \frac{-x}{x^2+y^2} - y\right) dy$